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Inst. of Computer Science, Univ. of Białystok

Linear Combinations in Vector Space

Wojciech A. Trybulec
Warsaw University

Summary. The notion of linear combination of vectors is introduced as a function from the carrier of a vector space to the carrier of the field. Definition of linear combination of set of vectors is also presented. We define addition and subtraction of combinations and multiplication of combination by element of the field. Sum of finite set of vectors and sum of linear combination is defined. We prove theorems that belong rather to [6].

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The articles [7], [12], [5], [2], [13], [3], [4], [8], [1], [9], [6], [10], and [11] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: x is a set, i is a natural number, G_1 is an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V is an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, v, v_1, v_2, v_3 are elements of V, a, b are elements of G_1, F, G are finite sequences of elements of the carrier of V, A, B are subsets of V, and f is a function from the carrier of V into the carrier of G_1.

Let G_1 be a non empty zero structure and let V be a non empty vector space structure over G_1. An element of (the carrier of G_1)$^{\text{the carrier of } V}$ is said to be a linear combination of V if:

(Def. 4)[1] There exists a finite subset T of V such that for every element v of V such that $v \notin T$ holds $\mathrm{it}(v) = 0_{(G_1)}$.

In the sequel L, L_1, L_2, L_3 denote linear combinations of V.

Let G_1 be a non empty zero structure, let V be a non empty vector space structure over G_1, and let L be a linear combination of V. The support of L yielding a finite subset of V is defined as follows:

(Def. 5) The support of $L = \{v; v$ ranges over elements of V: $L(v) \neq 0_{(G_1)}\}$.

We now state two propositions:

(19)[2] $x \in$ the support of L iff there exists v such that $x = v$ and $L(v) \neq 0_{(G_1)}$.

(20) $L(v) = 0_{(G_1)}$ iff $v \notin$ the support of L.

Let G_1 be a non empty zero structure and let V be a non empty vector space structure over G_1. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yields a linear combination of V and is defined by:

(Def. 6) The support of $\mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

Next we state the proposition

(22)[3] $\mathbf{0}_{\mathrm{LC}_V}(v) = 0_{(G_1)}$.

[1] The definitions (Def. 1)–(Def. 3) have been removed.
[2] The propositions (1)–(18) have been removed.
[3] The proposition (21) has been removed.

Let G_1 be a non empty zero structure, let V be a non empty vector space structure over G_1, and let A be a subset of V. A linear combination of V is said to be a linear combination of A if:

(Def. 7) The support of it $\subseteq A$.

In the sequel l is a linear combination of A.

Next we state four propositions:

(25)[4] If $A \subseteq B$, then l is a linear combination of B.

(26) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(27) For every linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(28) L is a linear combination of the support of L.

Let G_1 be a non empty loop structure, let V be a non empty vector space structure over G_1, let F be a finite sequence of elements of the carrier of V, and let f be a function from the carrier of V into the carrier of G_1. The functor $f\,F$ yielding a finite sequence of elements of the carrier of V is defined as follows:

(Def. 8) $\mathrm{len}(f\,F) = \mathrm{len}\,F$ and for every i such that $i \in \mathrm{dom}(f\,F)$ holds $(f\,F)(i) = f(F_i) \cdot F_i$.

We now state several propositions:

(32)[5] If $i \in \mathrm{dom}\,F$ and $v = F(i)$, then $(f\,F)(i) = f(v) \cdot v$.

(33) $f\,\varepsilon_{(\text{the carrier of } V)} = \varepsilon_{(\text{the carrier of } V)}$.

(34) $f\,\langle v \rangle = \langle f(v) \cdot v \rangle$.

(35) $f\,\langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(36) $f\,\langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

(37) $f\,(F \frown G) = (f\,F) \frown (f\,G)$.

Let G_1 be a non empty loop structure, let V be a non empty vector space structure over G_1, and let L be a linear combination of V. Let us assume that V is Abelian, add-associative, right zeroed, and right complementable. The functor $\sum L$ yields an element of V and is defined by the condition (Def. 9).

(Def. 9) There exists a finite sequence F of elements of the carrier of V such that F is one-to-one and $\mathrm{rng}\,F =$ the support of L and $\sum L = \sum(L\,F)$.

The following propositions are true:

(40)[6] If $0_{(G_1)} \neq \mathbf{1}_{(G_1)}$, then $A \neq \emptyset$ and A is linearly closed iff for every l holds $\sum l \in A$.

(41) $\sum(\mathbf{0}_{\mathrm{LC}_V}) = 0_V$.

(42) For every linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $\sum l = 0_V$.

(43) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(44) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(45) If the support of $L = \emptyset$, then $\sum L = 0_V$.

(46) If the support of $L = \{v\}$, then $\sum L = L(v) \cdot v$.

[4] The propositions (23) and (24) have been removed.
[5] The propositions (29)–(31) have been removed.
[6] The propositions (38) and (39) have been removed.

(47) If the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let G_1 be a non empty zero structure, let V be a non empty vector space structure over G_1, and let L_1, L_2 be linear combinations of V. Let us observe that $L_1 = L_2$ if and only if:

(Def. 10) For every element v of V holds $L_1(v) = L_2(v)$.

Let us consider G_1, let us consider V, and let us consider L_1, L_2. The functor $L_1 + L_2$ yields a linear combination of V and is defined by:

(Def. 11) For every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

We now state several propositions:

(51)[7] The support of $L_1 + L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(52) Suppose L_1 is a linear combination of A and L_2 is a linear combination of A. Then $L_1 + L_2$ is a linear combination of A.

(53) $L_1 + L_2 = L_2 + L_1$.

(54) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(55) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider G_1, let us consider V, a, and let us consider L. The functor $a \cdot L$ yielding a linear combination of V is defined by:

(Def. 12) For every v holds $(a \cdot L)(v) = a \cdot L(v)$.

One can prove the following propositions:

(58)[8] The support of $a \cdot L \subseteq$ the support of L.

(59) Let G_1 be a field, V be a vector space over G_1, a be an element of G_1, and L be a linear combination of V. If $a \neq 0_{(G_1)}$, then the support of $a \cdot L =$ the support of L.

(60) $0_{(G_1)} \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(61) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(62) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(63) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(64) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(65) $\mathbf{1}_{(G_1)} \cdot L = L$.

Let us consider G_1, let us consider V, and let us consider L. The functor $-L$ yielding a linear combination of V is defined as follows:

(Def. 13) $-L = (-\mathbf{1}_{(G_1)}) \cdot L$.

Let us note that the functor $-L$ is involutive.

The following four propositions are true:

(67)[9] $(-L)(v) = -L(v)$.

(68) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(69) The support of $-L =$ the support of L.

[7] The propositions (48)–(50) have been removed.
[8] The propositions (56) and (57) have been removed.
[9] The proposition (66) has been removed.

(70) If L is a linear combination of A, then $-L$ is a linear combination of A.

Let us consider G_1, let us consider V, and let us consider L_1, L_2. The functor $L_1 - L_2$ yielding a linear combination of V is defined as follows:

(Def. 14) $L_1 - L_2 = L_1 + -L_2$.

The following propositions are true:

(73)[10] $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(74) The support of $L_1 - L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(75) Suppose L_1 is a linear combination of A and L_2 is a linear combination of A. Then $L_1 - L_2$ is a linear combination of A.

(76) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

(77) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

(78) Let G_1 be a field, V be a vector space over G_1, L be a linear combination of V, and a be an element of G_1. Then $\sum(a \cdot L) = a \cdot \sum L$.

(79) $\sum(-L) = -\sum L$.

(80) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

(81) $(-\mathbf{1}_{(G_1)}) \cdot a = -a$.

(82) For every field G_1 holds $-\mathbf{1}_{(G_1)} \neq 0_{(G_1)}$.

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