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Operations on Subspaces in Vector Space

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Summary. Sum, direct sum and intersection of subspaces are introduced. We prove some theorems concerning those notions and the decomposition of vector onto two subspaces. Linear complement of a subspace is also defined. We prove theorems that belong rather to [4].

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The articles [6], [3], [9], [1], [10], [2], [12], [11], [7], [4], [5], and [8] provide the notation and terminology for this paper.

For simplicity, we use the following convention: G_1 denotes an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, M denotes an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, W, W_1, W_2, W_3 denote subspaces of M, u, v, v_1, v_2 denote elements of M, and x denotes a set.

Let us consider G_1, let us consider M, and let us consider W_1, W_2. The functor $W_1 + W_2$ yielding a strict subspace of M is defined by:

(Def. 1) The carrier of $W_1 + W_2 = \{v + u : v \in W_1 \ \wedge \ u \in W_2\}$.

Let us consider G_1, let us consider M, and let us consider W_1, W_2. The functor $W_1 \cap W_2$ yields a strict subspace of M and is defined as follows:

(Def. 2) The carrier of $W_1 \cap W_2 =$ (the carrier of W_1) $\cap$ (the carrier of W_2).

Let us note that the functor $W_1 \cap W_2$ is commutative.

The following propositions are true:

(5)[1] $x \in W_1 + W_2$ iff there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ iff $x \in W_1$ and $x \in W_2$.

(8) For every strict subspace W of M holds $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(12) For every strict subspace W_2 of M holds W_1 is a subspace of W_2 iff $W_1 + W_2 = W_2$.

[1] The propositions (1)–(4) have been removed.

(13) For every strict subspace W of M holds $\mathbf{0}_M + W = W$ and $W + \mathbf{0}_M = W$.

(14) $\mathbf{0}_M + \Omega_M =$ the vector space structure of M and $\Omega_M + \mathbf{0}_M =$ the vector space structure of M.

(15) $\Omega_M + W =$ the vector space structure of M and $W + \Omega_M =$ the vector space structure of M.

(16) Let M be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1. Then $\Omega_M + \Omega_M = M$.

(17) For every strict subspace W of M holds $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(21)(i) For every strict subspace W_1 of M such that W_1 is a subspace of W_2 holds $W_1 \cap W_2 = W_1$, and

(ii) for every W_1 such that $W_1 \cap W_2 = W_1$ holds W_1 is a subspace of W_2.

(22) If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(23) If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(24) If W_1 is a subspace of W_2 and a subspace of W_3, then W_1 is a subspace of $W_2 \cap W_3$.

(25) $\mathbf{0}_M \cap W = \mathbf{0}_M$ and $W \cap \mathbf{0}_M = \mathbf{0}_M$.

(27)[2] For every strict subspace W of M holds $\Omega_M \cap W = W$ and $W \cap \Omega_M = W$.

(28) Let M be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1. Then $\Omega_M \cap \Omega_M = M$.

(29) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(30) For every strict subspace W_2 of M holds $W_1 \cap W_2 + W_2 = W_2$.

(31) For every strict subspace W_1 of M holds $W_1 \cap (W_1 + W_2) = W_1$.

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(33) If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) For every strict subspace W_1 of M such that W_1 is a subspace of W_3 holds $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) For all strict subspaces W_1, W_2 of M holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(38) For all strict subspaces W_2, W_3 of M such that W_1 is a subspace of W_2 holds $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(39) If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(40) If W_1 is a subspace of W_3 and W_2 is a subspace of W_3, then $W_1 + W_2$ is a subspace of W_3.

(41) There exists W such that the carrier of $W =$ (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

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[2] The proposition (26) has been removed.

Let us consider G_1 and let us consider M. The functor Subspaces$M$ yields a set and is defined as follows:

(Def. 3) For every x holds $x \in$ Subspaces$M$ iff there exists a strict subspace W of M such that $W = x$.

Let us consider G_1 and let us consider M. One can check that Subspaces$M$ is non empty.

We now state the proposition

(44)[3] Let M be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1. Then $M \in$ Subspaces$M$.

Let us consider G_1, let us consider M, and let us consider W_1, W_2. We say that M is the direct sum of W_1 and W_2 if and only if:

(Def. 4) The vector space structure of $M = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_M$.

In the sequel F is a field, V is a vector space over F, and W is a subspace of V.

Let us consider F, V, W. A subspace of V is called a linear complement of W if:

(Def. 5) V is the direct sum of it and W.

In the sequel W, W_1, W_2 denote subspaces of V.

One can prove the following four propositions:

(47)[4] If V is the direct sum of W_1 and W_2, then W_2 is a linear complement of W_1.

(48) For every linear complement L of W holds V is the direct sum of L and W and the direct sum of W and L.

(49) Let L be a linear complement of W. Then $W + L =$ the vector space structure of V and $L + W =$ the vector space structure of V.

(50) For every linear complement L of W holds $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

In the sequel W_1, W_2 are subspaces of M.

The following two propositions are true:

(51) If M is the direct sum of W_1 and W_2, then M is the direct sum of W_2 and W_1.

(52) M is the direct sum of $\mathbf{0}_M$ and Ω_M and the direct sum of Ω_M and $\mathbf{0}_M$.

In the sequel W is a subspace of V.

Next we state two propositions:

(53) For every linear complement L of W holds W is a linear complement of L.

(54) $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

For simplicity, we adopt the following rules: W_1, W_2 denote subspaces of M, v denotes an element of M, C_1 denotes a coset of W_1, and C_2 denotes a coset of W_2.

Next we state several propositions:

(55) If C_1 meets C_2, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(56) M is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists an element v of M such that $C_1 \cap C_2 = \{v\}$.

(57) Let M be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1 and W_1, W_2 be subspaces of M. Then $W_1 + W_2 = M$ if and only if for every element v of M there exist elements v_1, v_2 of M such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

[3] The propositions (42) and (43) have been removed.

[4] The propositions (45) and (46) have been removed.

(58) Let v, v_1, v_2, u_1, u_2 be elements of M. Suppose M is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$. Then $v_1 = u_1$ and $v_2 = u_2$.

(59) Suppose that

(i) $M = W_1 + W_2$, and

(ii) there exists v such that for all elements v_1, v_2, u_1, u_2 of M such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$.

Then M is the direct sum of W_1 and W_2.

Let us consider G_1, M, v, W_1, W_2. Let us assume that M is the direct sum of W_1 and W_2. The functor $v_{\langle W_1, W_2 \rangle}$ yields an element of $[:\text{the carrier of } M, \text{the carrier of } M:]$ and is defined as follows:

(Def. 6) $v = (v_{\langle W_1, W_2 \rangle})_{\mathbf{1}} + (v_{\langle W_1, W_2 \rangle})_{\mathbf{2}}$ and $(v_{\langle W_1, W_2 \rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1, W_2 \rangle})_{\mathbf{2}} \in W_2$.

One can prove the following two propositions:

(64)[5] If M is the direct sum of W_1 and W_2, then $(v_{\langle W_1, W_2 \rangle})_{\mathbf{1}} = (v_{\langle W_2, W_1 \rangle})_{\mathbf{2}}$.

(65) If M is the direct sum of W_1 and W_2, then $(v_{\langle W_1, W_2 \rangle})_{\mathbf{2}} = (v_{\langle W_2, W_1 \rangle})_{\mathbf{1}}$.

In the sequel W is a subspace of V.
One can prove the following propositions:

(66) Let L be a linear complement of W, v be an element of V, and t be an element of $[:\text{the carrier of } V, \text{the carrier of } V:]$. If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v_{\langle W, L \rangle}$.

(67) For every linear complement L of W and for every element v of V holds $(v_{\langle W, L \rangle})_{\mathbf{1}} + (v_{\langle W, L \rangle})_{\mathbf{2}} = v$.

(68) For every linear complement L of W and for every element v of V holds $(v_{\langle W, L \rangle})_{\mathbf{1}} \in W$ and $(v_{\langle W, L \rangle})_{\mathbf{2}} \in L$.

(69) For every linear complement L of W and for every element v of V holds $(v_{\langle W, L \rangle})_{\mathbf{1}} = (v_{\langle L, W \rangle})_{\mathbf{2}}$.

(70) For every linear complement L of W and for every element v of V holds $(v_{\langle W, L \rangle})_{\mathbf{2}} = (v_{\langle L, W \rangle})_{\mathbf{1}}$.

In the sequel A_1, A_2 are elements of $\mathrm{Subspaces}\, M$ and W_1, W_2 are subspaces of M.
Let us consider G_1 and let us consider M. The functor $\mathrm{SubJoin}\, M$ yielding a binary operation on $\mathrm{Subspaces}\, M$ is defined as follows:

(Def. 7) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, M)(A_1, A_2) = W_1 + W_2$.

Let us consider G_1 and let us consider M. The functor $\mathrm{SubMeet}\, M$ yielding a binary operation on $\mathrm{Subspaces}\, M$ is defined as follows:

(Def. 8) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\, M)(A_1, A_2) = W_1 \cap W_2$.

We now state several propositions:

(75)[6] $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a lattice.

[5] The propositions (60)–(63) have been removed.
[6] The propositions (71)–(74) have been removed.

(76) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a lower bound lattice.

(77) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is an upper bound lattice.

(78) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a bound lattice.

(79) $\langle \mathrm{Subspaces}\, M, \mathrm{SubJoin}\, M, \mathrm{SubMeet}\, M \rangle$ is a modular lattice.

(80) For every field F and for every vector space V over F holds $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a complemented lattice.

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