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Families of Subsets, Subspaces and Mappings in Topological Spaces

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Summary. This article is a continuation of [11]. Some basic theorems about families of sets in a topological space have been proved. Following redefinitions have been made: singleton of a set as a family in the topological space and results of boolean operations on families as a family of the topological space. Notion of a family of complements of sets and a closed (open) family have been also introduced. Next some theorems refer to subspaces in a topological space: some facts about types in a subspace, theorems about open and closed sets and families in a subspace. A notion of restriction of a family has been also introduced and basic properties of this notion have been proved. The last part of the article is about mappings. There are proved necessary and sufficient conditions for a mapping to be continuous. A notion of homeomorphism has been defined next. Theorems about homeomorphisms of topological spaces have been also proved.

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The articles [8], [4], [9], [10], [2], [3], [1], [5], [6], and [7] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x is a set, T is a topological structure, G_1 is a topological space, P, Q are subsets of T, M, N are subsets of T, F, G are families of subsets of T, W is a family of subsets of G_1, and A is a subspace of T.

One can prove the following propositions:

(1) For every 1-sorted structure T and for every family F of subsets of T holds $F \subseteq 2^{\Omega_T}$.

(3)[1] Let T be a 1-sorted structure, F be a family of subsets of T, and X be a set. If $X \subseteq F$, then X is a family of subsets of T.

(5)[2] For every non empty 1-sorted structure T and for every family F of subsets of T such that F is a cover of T holds $F \neq \emptyset$.

(6) For every 1-sorted structure T and for all families F, G of subsets of T holds $\bigcup F \setminus \bigcup G \subseteq \bigcup (F \setminus G)$.

(9)[3] For every 1-sorted structure T and for every family F of subsets of T holds $(F^{\rm c})^{\rm c} = F$.

(10) For every 1-sorted structure T and for every family F of subsets of T holds $F \neq \emptyset$ iff $F^{\rm c} \neq \emptyset$.

[1] The proposition (2) has been removed.
[2] The proposition (4) has been removed.
[3] The propositions (7) and (8) have been removed.

(11) For every 1-sorted structure T and for every family F of subsets of T such that $F \neq \emptyset$ holds $\bigcap(F^{\rm c}) = (\bigcup F)^{\rm c}$.

(12) For every 1-sorted structure T and for every family F of subsets of T such that $F \neq \emptyset$ holds $\bigcup(F^{\rm c}) = (\bigcap F)^{\rm c}$.

(13) Let T be a 1-sorted structure and F be a family of subsets of T. Then $F^{\rm c}$ is finite if and only if F is finite.

Let T be a topological structure and let F be a family of subsets of T. We say that F is open if and only if:

(Def. 1) For every subset P of T such that $P \in F$ holds P is open.

We say that F is closed if and only if:

(Def. 2) For every subset P of T such that $P \in F$ holds P is closed.

We now state a number of propositions:

(16)[4] F is closed iff $F^{\rm c}$ is open.

(17) F is open iff $F^{\rm c}$ is closed.

(18) If $F \subseteq G$ and G is open, then F is open.

(19) If $F \subseteq G$ and G is closed, then F is closed.

(20) If F is open and G is open, then $F \cup G$ is open.

(21) If F is open, then $F \cap G$ is open.

(22) If F is open, then $F \setminus G$ is open.

(23) If F is closed and G is closed, then $F \cup G$ is closed.

(24) If F is closed, then $F \cap G$ is closed.

(25) If F is closed, then $F \setminus G$ is closed.

(26) If W is open, then $\bigcup W$ is open.

(27) If W is open and finite, then $\bigcap W$ is open.

(28) If W is closed and finite, then $\bigcup W$ is closed.

(29) If W is closed, then $\bigcap W$ is closed.

(31)[5] Every family of subsets of A is a family of subsets of T.

(32) For every subset B of A holds B is open iff there exists a subset C of T such that C is open and $C \cap \Omega_A = B$.

(33) If Q is open, then for every subset P of A such that $P = Q$ holds P is open.

(34) If Q is closed, then for every subset P of A such that $P = Q$ holds P is closed.

(35) If F is open, then for every family G of subsets of A such that $G = F$ holds G is open.

(36) If F is closed, then for every family G of subsets of A such that $G = F$ holds G is closed.

(38)[6] $M \cap N$ is a subset of $T \restriction N$.

[4] The propositions (14) and (15) have been removed.
[5] The proposition (30) has been removed.
[6] The proposition (37) has been removed.

Let T be a topological structure, let P be a subset of T, and let F be a family of subsets of T. The functor $F\restriction P$ yielding a family of subsets of $T\restriction P$ is defined as follows:

(Def. 3) For every subset Q of $T\restriction P$ holds $Q \in F\restriction P$ iff there exists a subset R of T such that $R \in F$ and $R \cap P = Q$.

The following propositions are true:

(40)[7] If $F \subseteq G$, then $F\restriction M \subseteq G\restriction M$.

(41) If $Q \in F$, then $Q \cap M \in F\restriction M$.

(42) If $Q \subseteq \bigcup F$, then $Q \cap M \subseteq \bigcup(F\restriction M)$.

(43) If $M \subseteq \bigcup F$, then $M = \bigcup(F\restriction M)$.

(44) $\bigcup(F\restriction M) \subseteq \bigcup F$.

(45) If $M \subseteq \bigcup(F\restriction M)$, then $M \subseteq \bigcup F$.

(46) If F is finite, then $F\restriction M$ is finite.

(47) If F is open, then $F\restriction M$ is open.

(48) If F is closed, then $F\restriction M$ is closed.

(49) Let F be a family of subsets of A. Suppose F is open. Then there exists a family G of subsets of T such that G is open and for every subset A_1 of T such that $A_1 = \Omega_A$ holds $F = G\restriction A_1$.

(50) There exists a function f such that $\operatorname{dom} f = F$ and $\operatorname{rng} f = F\restriction P$ and for every x such that $x \in F$ and for every Q such that $Q = x$ holds $f(x) = Q \cap P$.

(51) Let T be a 1-sorted structure, S be a non empty 1-sorted structure, and f be a map from T into S. Then $\operatorname{dom} f = \Omega_T$ and $\operatorname{rng} f \subseteq \Omega_S$.

(52) Let T be a 1-sorted structure, S be a non empty 1-sorted structure, and f be a map from T into S. Then $f^{-1}(\Omega_S) = \Omega_T$.

(54)[8] Let T be a 1-sorted structure, S be a non empty 1-sorted structure, f be a map from T into S, and H be a family of subsets of S. Then $(^{-1}f)^{\circ}H$ is a family of subsets of T.

For simplicity, we follow the rules: S, V are non empty topological structures, f is a map from T into S, g is a map from S into V, H is a family of subsets of S, and P_1 is a subset of S.

The following propositions are true:

(55) f is continuous iff for every P_1 such that P_1 is open holds $f^{-1}(P_1)$ is open.

(56) Let T be a topological space, S be a non empty topological space, and f be a map from T into S. Then f is continuous if and only if for every subset P_1 of S holds $\overline{f^{-1}(P_1)} \subseteq f^{-1}(\overline{P_1})$.

(57) Let T be a topological space, S be a non empty topological space, and f be a map from T into S. Then f is continuous if and only if for every subset P of T holds $f^{\circ}\overline{P} \subseteq \overline{f^{\circ}P}$.

(58) If f is continuous and g is continuous, then $g \cdot f$ is continuous.

(59) If f is continuous and H is open, then for every F such that $F = (^{-1}f)^{\circ}H$ holds F is open.

(60) Let T, S be topological structures, f be a map from T into S, and H be a family of subsets of S. Suppose f is continuous and H is closed. Let F be a family of subsets of T. If $F = (^{-1}f)^{\circ}H$, then F is closed.

[7] The proposition (39) has been removed.

[8] The proposition (53) has been removed.

Let T, S be 1-sorted structures and let f be a map from T into S. Let us assume that $\operatorname{rng} f = \Omega_S$ and f is one-to-one. The functor UNKNOWN(f) yielding a map from S into T is defined by:

(Def. 4) UNKNOWN$(f) = f^{-1}$.

We introduce f^{-1} as a synonym of UNKNOWN(f).

Next we state several propositions:

(62)[9] Let T be a 1-sorted structure, S be a non empty 1-sorted structure, and f be a map from T into S. If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $\operatorname{dom}(f^{-1}) = \Omega_S$ and $\operatorname{rng}(f^{-1}) = \Omega_T$.

(63) Let T, S be 1-sorted structures and f be a map from T into S. If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then f^{-1} is one-to-one.

(64) Let T be a 1-sorted structure, S be a non empty 1-sorted structure, and f be a map from T into S. If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $(f^{-1})^{-1} = f$.

(65) Let T, S be 1-sorted structures and f be a map from T into S. If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(66) Let T be a 1-sorted structure, S, V be non empty 1-sorted structures, f be a map from T into S, and g be a map from S into V. Suppose $\operatorname{dom} f = \Omega_T$ and $\operatorname{rng} f = \Omega_S$ and f is one-to-one and $\operatorname{dom} g = \Omega_S$ and $\operatorname{rng} g = \Omega_V$ and g is one-to-one. Then $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

(67) Let T, S be 1-sorted structures, f be a map from T into S, and P be a subset of T. If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{\circ}P = (f^{-1})^{-1}(P)$.

(68) Let T, S be 1-sorted structures, f be a map from T into S, and P_1 be a subset of S. If $\operatorname{rng} f = \Omega_S$ and f is one-to-one, then $f^{-1}(P_1) = (f^{-1})^{\circ}P_1$.

Let S, T be topological structures and let f be a map from S into T. We say that f is homeomorphism if and only if:

(Def. 5) $\operatorname{dom} f = \Omega_S$ and $\operatorname{rng} f = \Omega_T$ and f is one-to-one and continuous and f^{-1} is continuous.

We introduce f is a homeomorphism as a synonym of f is homeomorphism.

One can prove the following three propositions:

(70)[10] If f is a homeomorphism, then f^{-1} is a homeomorphism.

(71) Let T, S, V be non empty topological structures, f be a map from T into S, and g be a map from S into V. If f is a homeomorphism and g is a homeomorphism, then $g \cdot f$ is a homeomorphism.

(72) f is a homeomorphism if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f = \Omega_T$,

(ii) $\operatorname{rng} f = \Omega_S$,

(iii) f is one-to-one, and

(iv) for every P holds P is closed iff $f^{\circ}P$ is closed.

For simplicity, we adopt the following convention: T, S are non empty topological spaces, P is a subset of T, P_1 is a subset of S, and f is a map from T into S.

The following propositions are true:

(73) f is a homeomorphism iff $\operatorname{dom} f = \Omega_T$ and $\operatorname{rng} f = \Omega_S$ and f is one-to-one and for every P_1 holds $f^{-1}(\overline{P_1}) = \overline{f^{-1}(P_1)}$.

(74) f is a homeomorphism iff $\operatorname{dom} f = \Omega_T$ and $\operatorname{rng} f = \Omega_S$ and f is one-to-one and for every P holds $f^{\circ}\overline{P} = \overline{f^{\circ}P}$.

[9] The proposition (61) has been removed.

[10] The proposition (69) has been removed.

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