JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Topological Space $\mathcal{E}^2_{\rm T}$. Arcs, Line Segments and Special Polygonal Arcs

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Summary. The notions of arc and line segment are introduced in two-dimensional topological real space $\mathcal{E}^2_{\rm T}$. Some basic theorems for these notions are proved. Using line segments, the notion of special polygonal arc is defined. It has been shown that any special polygonal arc is homeomorphic to unit interval $\mathbb{I}$. The notion of unit square $\square_{\mathcal{E}^{\in}_{\rm T}}$ has been also introduced and some facts about it have been proved.

MML Identifier: TOPREAL1.

WWW: http://mizar.org/JFM/Vol3/topreal1.html

The articles [15], [18], [2], [3], [16], [11], [1], [19], [6], [7], [8], [13], [4], [17], [12], [10], [5], [9], and [14] provide the notation and terminology for this paper.

We adopt the following rules: l_1 denotes a real number, i, j, n denote natural numbers, and a, m denote natural numbers.

The scheme *Fraenkel Alt* deals with a non empty set $\mathcal{A}$ and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{v; v$ ranges over elements of $\mathcal{A} : \mathcal{P}[v] \ \vee \ \mathcal{Q}[v]\} = \{v_1; v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} \cup \{v_2; v_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[v_2]\}$

for all values of the parameters.

In the sequel D denotes a set and p denotes a finite sequence of elements of D.

Let us consider D, p, m. The functor $p{\restriction}m$ yielding a finite sequence of elements of D is defined by:

(Def. 1) $p{\restriction}m = p{\restriction}\operatorname{Seg} m$.

Let D be a set and let f be a finite sequence of elements of D. One can verify that $f{\restriction}0$ is empty.

The following propositions are true:

(1) If $a \in \operatorname{dom}(p{\restriction}m)$, then $(p{\restriction}m)_a = p_a$.

(2) If $\operatorname{len} p \leq m$, then $p{\restriction}m = p$.

(3) If $m \leq \operatorname{len} p$, then $\operatorname{len}(p{\restriction}m) = m$.

Let T be a 1-sorted structure. A finite sequence of elements of T is a finite sequence of elements of the carrier of T.

We use the following convention: p, p_1, p_2, q_1, q_2 are points of $\mathcal{E}^2_{\rm T}$ and P, P_1 are subsets of $\mathcal{E}^2_{\rm T}$.

Let us consider n, let p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$, and let P be a subset of $\mathcal{E}^n_{\rm T}$. We say that P is an arc from p_1 to p_2 if and only if:

(Def. 2) There exists a map f from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$.

Next we state two propositions:

(4) For every subset P of $\mathcal{E}^n_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds $p_1 \in P$ and $p_2 \in P$.

(5) Let P, Q be subsets of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and Q is an arc from p_2 to q_1 and $P \cap Q = \{p_2\}$. Then $P \cup Q$ is an arc from p_1 to q_1.

The subset $\square_{\mathcal{E}^2}$ of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def. 3).

(Def. 3) $\square_{\mathcal{E}^2} = \{p : p_{\mathbf{1}} = 0 \ \wedge\ p_{\mathbf{2}} \leq 1 \ \wedge\ p_{\mathbf{2}} \geq 0 \ \vee\ p_{\mathbf{1}} \leq 1 \ \wedge\ p_{\mathbf{1}} \geq 0 \ \wedge\ p_{\mathbf{2}} = 1 \ \vee\ p_{\mathbf{1}} \leq 1 \ \wedge\ p_{\mathbf{1}} \geq 0 \ \wedge\ p_{\mathbf{2}} = 0 \ \vee\ p_{\mathbf{1}} = 1 \ \wedge\ p_{\mathbf{2}} \leq 1 \ \wedge\ p_{\mathbf{2}} \geq 0\}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathcal{L}(p_1, p_2)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 4) $\mathcal{L}(p_1, p_2) = \{(1 - l_1) \cdot p_1 + l_1 \cdot p_2 : 0 \leq l_1 \ \wedge\ l_1 \leq 1\}$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Note that $\mathcal{L}(p_1, p_2)$ is non empty.

The following propositions are true:

(6) For all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ holds $p_1 \in \mathcal{L}(p_1, p_2)$ and $p_2 \in \mathcal{L}(p_1, p_2)$.

(7) For every point p of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathcal{L}(p, p) = \{p\}$.

(8) For all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_2, p_1)$.

Let us consider n and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us note that the functor $\mathcal{L}(p_1, p_2)$ is commutative.

Next we state a number of propositions:

(9) If $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$ and $p \in \mathcal{L}(p_1, p_2)$, then $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$.

(10) If $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$ and $p \in \mathcal{L}(p_1, p_2)$, then $(p_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$.

(11) For all points p, p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in \mathcal{L}(p_1, p_2)$ holds $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_1, p) \cup \mathcal{L}(p, p_2)$.

(12) For all points p_1, p_2, q_1, q_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$ holds $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$.

(13) For all points p, q, p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in \mathcal{L}(p_1, p_2)$ and $q \in \mathcal{L}(p_1, p_2)$ holds $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_1, p) \cup \mathcal{L}(p, q) \cup \mathcal{L}(q, p_2)$.

(14) If $p \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p) \cap \mathcal{L}(p, p_2) = \{p\}$.

(15) For all points p_1, p_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that $p_1 \neq p_2$ holds $\mathcal{L}(p_1, p_2)$ is an arc from p_1 to p_2.

(16) Let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $P \cap \mathcal{L}(p_2, q_1) = \{p_2\}$, then $P \cup \mathcal{L}(p_2, q_1)$ is an arc from p_1 to q_1.

(17) Let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^n_{\mathrm{T}}$. If P is an arc from p_2 to p_1 and $\mathcal{L}(q_1, p_2) \cap P = \{p_2\}$, then $\mathcal{L}(q_1, p_2) \cup P$ is an arc from q_1 to p_1.

(18) For all points p_1, p_2, q_1 of $\mathcal{E}^n_{\mathrm{T}}$ such that $p_1 \neq p_2$ or $p_2 \neq q_1$ but $\mathcal{L}(p_1, p_2) \cap \mathcal{L}(p_2, q_1) = \{p_2\}$ holds $\mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, q_1)$ is an arc from p_1 to q_1.

(19)(i) $\mathcal{L}([0, 0], [0, 1]) = \{p_1 : (p_1)_{\mathbf{1}} = 0 \ \wedge\ (p_1)_{\mathbf{2}} \leq 1 \ \wedge\ (p_1)_{\mathbf{2}} \geq 0\}$,

(ii) $\mathcal{L}([0, 1], [1, 1]) = \{p_2 : (p_2)_{\mathbf{1}} \leq 1 \ \wedge\ (p_2)_{\mathbf{1}} \geq 0 \ \wedge\ (p_2)_{\mathbf{2}} = 1\}$,

(iii) $\mathcal{L}([0, 0], [1, 0]) = \{q_1 : (q_1)_{\mathbf{1}} \leq 1 \ \wedge\ (q_1)_{\mathbf{1}} \geq 0 \ \wedge\ (q_1)_{\mathbf{2}} = 0\}$, and

(iv) $\mathcal{L}([1, 0], [1, 1]) = \{q_2 : (q_2)_{\mathbf{1}} = 1 \ \wedge\ (q_2)_{\mathbf{2}} \leq 1 \ \wedge\ (q_2)_{\mathbf{2}} \geq 0\}$.

(20) $\square_{\mathcal{E}^2} = \mathcal{L}([0,0],[0,1]) \cup \mathcal{L}([0,1],[1,1]) \cup (\mathcal{L}([0,0],[1,0]) \cup \mathcal{L}([1,0],[1,1]))$.

Let us note that $\square_{\mathcal{E}^2}$ is non empty.
We now state several propositions:

(21) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([0,1],[1,1]) = \{[0,1]\}$.

(22) $\mathcal{L}([0,0],[1,0]) \cap \mathcal{L}([1,0],[1,1]) = \{[1,0]\}$.

(23) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([0,0],[1,0]) = \{[0,0]\}$.

(24) $\mathcal{L}([0,1],[1,1]) \cap \mathcal{L}([1,0],[1,1]) = \{[1,1]\}$.

(25) $\mathcal{L}([0,0],[1,0])$ misses $\mathcal{L}([0,1],[1,1])$.

(26) $\mathcal{L}([0,0],[0,1])$ misses $\mathcal{L}([1,0],[1,1])$.

Let us consider n, let f be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$, and let us consider i. The functor $\mathcal{L}(f,i)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 5) $\mathcal{L}(f,i) = \begin{cases} \mathcal{L}(f_i, f_{i+1}), & \text{if } 1 \leq i \text{ and } i+1 \leq \operatorname{len} f, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(27) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ holds $f_i \in \mathcal{L}(f,i)$ and $f_{i+1} \in \mathcal{L}(f,i)$.

Let us consider n and let f be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\widetilde{\mathcal{L}}(f)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 6) $\widetilde{\mathcal{L}}(f) = \bigcup\{\mathcal{L}(f,i) : 1 \leq i \wedge i+1 \leq \operatorname{len} f\}$.

The following propositions are true:

(28) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ holds $\operatorname{len} f = 0$ or $\operatorname{len} f = 1$ iff $\widetilde{\mathcal{L}}(f) = \emptyset$.

(29) For every finite sequence f of elements of $\mathcal{E}^n_{\mathrm{T}}$ such that $\operatorname{len} f \geq 2$ holds $\widetilde{\mathcal{L}}(f) \neq \emptyset$.

Let I_1 be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. We say that I_1 is special if and only if:

(Def. 7) For every i such that $1 \leq i$ and $i+1 \leq \operatorname{len} I_1$ holds $((I_1)_i)_{\mathbf{1}} = ((I_1)_{i+1})_{\mathbf{1}}$ or $((I_1)_i)_{\mathbf{2}} = ((I_1)_{i+1})_{\mathbf{2}}$.

We say that I_1 is unfolded if and only if:

(Def. 8) For every i such that $1 \leq i$ and $i+2 \leq \operatorname{len} I_1$ holds $\mathcal{L}(I_1,i) \cap \mathcal{L}(I_1,i+1) = \{(I_1)_{i+1}\}$.

We say that I_1 is s.n.c. if and only if:

(Def. 9) For all i, j such that $i+1 < j$ holds $\mathcal{L}(I_1,i)$ misses $\mathcal{L}(I_1,j)$.

In the sequel f, f_1, f_2, h are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$.
Let us consider f. We say that f is special sequence if and only if:

(Def. 10) f is one-to-one and $\operatorname{len} f \geq 2$ and f is unfolded, s.n.c., and special.

We introduce f is a special sequence as a synonym of f is special sequence.
One can prove the following two propositions:

(30) There exist f_1, f_2 such that
f_1 is a special sequence and f_2 is a special sequence and $\square_{\mathcal{E}^2} = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{[0,0],[1,1]\}$ and $(f_1)_1 = [0,0]$ and $(f_1)_{\operatorname{len} f_1} = [1,1]$ and $(f_2)_1 = [0,0]$ and $(f_2)_{\operatorname{len} f_2} = [1,1]$.

(31) If h is a special sequence, then $\widetilde{\mathcal{L}}(h)$ is an arc from h_1 to $h_{\mathrm{len}\, h}$.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. We say that P is special polygonal arc if and only if:

(Def. 11) There exists f such that f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$.

We introduce P is a special polygonal arc as a synonym of P is special polygonal arc.
Next we state the proposition

(32) If P_1 is a special polygonal arc, then $P_1 \neq \emptyset$.

One can verify that every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is special polygonal arc is also non empty.
We now state three propositions:

(34)[1] There exist non empty subsets P_1, P_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P_1 is a special polygonal arc and P_2 is a special polygonal arc and $\square_{\mathcal{E}^2} = P_1 \cup P_2$ and $P_1 \cap P_2 = \{[0,0],[1,1]\}$.

(35) If P is a special polygonal arc, then there exist p_1, p_2 such that P is an arc from p_1 to p_2.

(36) If P is a special polygonal arc, then there exists a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}} \restriction P$ which is a homeomorphism.

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Received November 21, 1991

Published January 2, 2004