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On Nowhere and Everywhere Dense Subspaces of Topological Spaces

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Summary. Let X be a topological space and let X_0 be a subspace of X with the carrier A. X_0 is called *boundary* (*dense*) in X if A is boundary (dense), i.e., $\operatorname{Int} A = \emptyset$ ($\overline{A} =$ the carrier of X); X_0 is called *nowhere dense* (*everywhere dense*) in X if A is nowhere dense (everywhere dense), i.e., $\operatorname{Int}\overline{A} = \emptyset$ ($\overline{\operatorname{Int} A} =$ the carrier of X) (see [6] and comp. [7]).

Our purpose is to list, using Mizar formalism, a number of properties of such subspaces, mostly in non-discrete (non-almost-discrete) spaces (comp. [6]). Recall that X is called *discrete* if every subset of X is open (closed); X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open (see [1], [5] and comp. [7],[8]). We have the following characterization of non-discrete spaces: *X is non-discrete iff there exists a boundary subspace in X*. Hence, *X is non-discrete iff there exists a dense proper subspace in X*. We have the following analogous characterization of non-almost-discrete spaces: *X is non-almost-discrete iff there exists a nowhere dense subspace in X*. Hence, *X is non-almost-discrete iff there exists an everywhere dense proper subspace in X*.

Note that some interdependencies between boundary, dense, nowhere and everywhere dense subspaces are also indicated. These have the form of observations in the text and they correspond to the existential and to the conditional clusters in the Mizar System. These clusters guarantee the existence and ensure the extension of types supported automatically by the Mizar System.

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The articles [10], [12], [9], [13], [11], [3], [2], [1], [6], and [4] provide the notation and terminology for this paper.

1. Some Properties of Subsets of a Topological Space

In this paper X denotes a non empty topological space and A, B denote subsets of X.

Next we state several propositions:

(1) If A and B constitute a decomposition, then A is non empty iff B is proper.

(2) If A and B constitute a decomposition, then A is dense iff B is boundary.

(3) If A and B constitute a decomposition, then A is boundary iff B is dense.

(4) If A and B constitute a decomposition, then A is everywhere dense iff B is nowhere dense.

(5) If A and B constitute a decomposition, then A is nowhere dense iff B is everywhere dense.

In the sequel Y_1, Y_2 denote non empty subspaces of X.
One can prove the following propositions:

(6) If Y_1 and Y_2 constitute a decomposition, then Y_1 is proper and Y_2 is proper.

(7) Let X be a non trivial non empty topological space and D be a non empty proper subset of X. Then there exists a proper strict non empty subspace Y_0 of X such that $D =$ the carrier of Y_0.

(8) Let X be a non trivial non empty topological space and Y_1 be a proper non empty subspace of X. Then there exists a proper strict non empty subspace Y_2 of X such that Y_1 and Y_2 constitute a decomposition.

2. Dense and Everywhere Dense Subspaces

Let X be a non empty topological space and let I_1 be a subspace of X. We say that I_1 is dense if and only if:

(Def. 1) For every subset A of X such that $A =$ the carrier of I_1 holds A is dense.

We now state the proposition

(9) Let X_0 be a subspace of X and A be a subset of X. If $A =$ the carrier of X_0, then X_0 is dense iff A is dense.

Let X be a non empty topological space. One can check the following observations:

* every subspace of X which is dense and closed is also non proper,
* every subspace of X which is dense and proper is also non closed, and
* every subspace of X which is proper and closed is also non dense.

Let X be a non empty topological space. Note that there exists a subspace of X which is dense, strict, and non empty.

We now state several propositions:

(10) Let A_0 be a non empty subset of X. Suppose A_0 is dense. Then there exists a dense strict non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(11) Let X_0 be a dense non empty subspace of X, A be a subset of X, and B be a subset of X_0. If $A = B$, then B is dense iff A is dense.

(12) For every dense subspace X_1 of X and for every subspace X_2 of X such that X_1 is a subspace of X_2 holds X_2 is dense.

(13) Let X_1 be a dense non empty subspace of X and X_2 be a non empty subspace of X. If X_1 is a subspace of X_2, then X_1 is a dense subspace of X_2.

(14) For every dense non empty subspace X_1 of X holds every dense non empty subspace of X_1 is a dense non empty subspace of X.

(15) Let Y_1, Y_2 be non empty topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is a dense subspace of X if and only if Y_2 is a dense subspace of X.

Let X be a non empty topological space and let I_1 be a subspace of X. We say that I_1 is everywhere dense if and only if:

(Def. 2) For every subset A of X such that $A =$ the carrier of I_1 holds A is everywhere dense.

The following proposition is true

(16) Let X_0 be a subspace of X and A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is everywhere dense if and only if A is everywhere dense.

Let X be a non empty topological space. One can verify the following observations:

* every subspace of X which is everywhere dense is also dense,
* every subspace of X which is non dense is also non everywhere dense,
* every subspace of X which is non proper is also everywhere dense, and
* every subspace of X which is non everywhere dense is also proper.

Let X be a non empty topological space. Observe that there exists a subspace of X which is everywhere dense, strict, and non empty.

We now state several propositions:

(17) Let A_0 be a non empty subset of X. Suppose A_0 is everywhere dense. Then there exists an everywhere dense strict non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(18) Let X_0 be an everywhere dense non empty subspace of X, A be a subset of X, and B be a subset of X_0. Suppose $A = B$. Then B is everywhere dense if and only if A is everywhere dense.

(19) Let X_1 be an everywhere dense subspace of X and X_2 be a subspace of X. If X_1 is a subspace of X_2, then X_2 is everywhere dense.

(20) Let X_1 be an everywhere dense non empty subspace of X and X_2 be a non empty subspace of X. Suppose X_1 is a subspace of X_2. Then X_1 is an everywhere dense subspace of X_2.

(21) For every everywhere dense non empty subspace X_1 of X holds every everywhere dense non empty subspace of X_1 is an everywhere dense subspace of X.

(22) Let Y_1, Y_2 be non empty topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is an everywhere dense subspace of X if and only if Y_2 is an everywhere dense subspace of X.

Let X be a non empty topological space. One can verify the following observations:

* every subspace of X which is dense and open is also everywhere dense,
* every subspace of X which is dense and non everywhere dense is also non open, and
* every subspace of X which is open and non everywhere dense is also non dense.

Let X be a non empty topological space. Note that there exists a subspace of X which is dense, open, strict, and non empty.

Next we state two propositions:

(23) Let A_0 be a non empty subset of X. Suppose A_0 is dense and open. Then there exists a dense open strict non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(24) For every subspace X_0 of X holds X_0 is everywhere dense iff there exists a dense open strict subspace of X which is a subspace of X_0.

In the sequel X_1, X_2 are non empty subspaces of X.

One can prove the following propositions:

(25) If X_1 is dense or X_2 is dense, then $X_1 \cup X_2$ is a dense subspace of X.

(26) If X_1 is everywhere dense or X_2 is everywhere dense, then $X_1 \cup X_2$ is an everywhere dense subspace of X.

(27) If X_1 is everywhere dense and X_2 is everywhere dense, then $X_1 \cap X_2$ is an everywhere dense subspace of X.

(28) Suppose X_1 is everywhere dense and X_2 is dense or X_1 is dense and X_2 is everywhere dense. Then $X_1 \cap X_2$ is a dense subspace of X.

3. Boundary and Nowhere Dense Subspaces

Let X be a non empty topological space and let I_1 be a subspace of X. We say that I_1 is boundary if and only if:

(Def. 3) For every subset A of X such that $A =$ the carrier of I_1 holds A is boundary.

Next we state the proposition

(29) Let X_0 be a subspace of X and A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is boundary if and only if A is boundary.

Let X be a non empty topological space. One can verify the following observations:

* every non empty subspace of X which is open is also non boundary,
* every non empty subspace of X which is boundary is also non open,
* every subspace of X which is everywhere dense is also non boundary, and
* every subspace of X which is boundary is also non everywhere dense.

The following propositions are true:

(30) Let A_0 be a non empty subset of X. Suppose A_0 is boundary. Then there exists a strict subspace X_0 of X such that X_0 is boundary and $A_0 =$ the carrier of X_0.

(31) Let X_1, X_2 be subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is dense if and only if X_2 is boundary.

(32) Let X_1, X_2 be non empty subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is boundary if and only if X_2 is dense.

(33) Let X_0 be a subspace of X. Suppose X_0 is boundary. Let A be a subset of X. If $A \subseteq$ the carrier of X_0, then A is boundary.

(34) For all subspaces X_1, X_2 of X such that X_1 is boundary holds if X_2 is a subspace of X_1, then X_2 is boundary.

Let X be a non empty topological space and let I_1 be a subspace of X. We say that I_1 is nowhere dense if and only if:

(Def. 4) For every subset A of X such that $A =$ the carrier of I_1 holds A is nowhere dense.

One can prove the following proposition

(35) Let X_0 be a subspace of X and A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is nowhere dense if and only if A is nowhere dense.

Let X be a non empty topological space. One can verify the following observations:

* every subspace of X which is nowhere dense is also boundary,
* every subspace of X which is non boundary is also non nowhere dense,
* every subspace of X which is nowhere dense is also non dense, and
* every subspace of X which is dense is also non nowhere dense.

In the sequel X denotes a non empty topological space.
The following propositions are true:

(36) Let A_0 be a non empty subset of X. Suppose A_0 is nowhere dense. Then there exists a strict subspace X_0 of X such that X_0 is nowhere dense and $A_0 =$ the carrier of X_0.

(37) Let X_1, X_2 be subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is everywhere dense if and only if X_2 is nowhere dense.

(38) Let X_1, X_2 be non empty subspaces of X. Suppose X_1 and X_2 constitute a decomposition. Then X_1 is nowhere dense if and only if X_2 is everywhere dense.

(39) Let X_0 be a subspace of X. Suppose X_0 is nowhere dense. Let A be a subset of X. If $A \subseteq$ the carrier of X_0, then A is nowhere dense.

(40) Let X_1, X_2 be subspaces of X. Suppose X_1 is nowhere dense. If X_2 is a subspace of X_1, then X_2 is nowhere dense.

Let X be a non empty topological space. One can check the following observations:

* every subspace of X which is boundary and closed is also nowhere dense,
* every subspace of X which is boundary and non nowhere dense is also non closed, and
* every subspace of X which is closed and non nowhere dense is also non boundary.

Next we state two propositions:

(41) Let A_0 be a non empty subset of X. Suppose A_0 is boundary and closed. Then there exists a closed strict non empty subspace X_0 of X such that X_0 is boundary and $A_0 =$ the carrier of X_0.

(42) Let X_0 be a non empty subspace of X. Then X_0 is nowhere dense if and only if there exists a closed strict non empty subspace X_1 of X such that X_1 is boundary and X_0 is a subspace of X_1.

In the sequel X_1, X_2 denote non empty subspaces of X.
The following propositions are true:

(43) If X_1 is boundary or X_2 is boundary and if X_1 meets X_2, then $X_1 \cap X_2$ is boundary.

(44) If X_1 is nowhere dense and X_2 is nowhere dense, then $X_1 \cup X_2$ is nowhere dense.

(45) If X_1 is nowhere dense and X_2 is boundary or X_1 is boundary and X_2 is nowhere dense, then $X_1 \cup X_2$ is boundary.

(46) If X_1 is nowhere dense or X_2 is nowhere dense and if X_1 meets X_2, then $X_1 \cap X_2$ is nowhere dense.

4. Dense and Boundary Subspaces of Non-discrete Spaces

One can prove the following two propositions:

(47) For every non empty topological space X such that every subspace of X is non boundary holds X is discrete.

(48) For every non trivial non empty topological space X such that every proper subspace of X is non dense holds X is discrete.

Let X be a discrete non empty topological space. One can verify the following observations:

* every non empty subspace of X is non boundary,
* every subspace of X which is proper is also non dense, and
* every subspace of X which is dense is also non proper.

Let X be a discrete non empty topological space. One can check that there exists a subspace of X which is non boundary, strict, and non empty.

Let X be a discrete non trivial non empty topological space. Observe that there exists a subspace of X which is non dense and strict.

We now state two propositions:

(49) For every non empty topological space X such that there exists a non empty subspace of X which is boundary holds X is non discrete.

(50) For every non empty topological space X such that there exists a non empty subspace of X which is dense and proper holds X is non discrete.

Let X be a non discrete non empty topological space. One can check that there exists a subspace of X which is boundary, strict, and non empty and there exists a subspace of X which is dense, proper, strict, and non empty.

In the sequel X denotes a non discrete non empty topological space.

We now state several propositions:

(51) Let A_0 be a non empty subset of X. Suppose A_0 is boundary. Then there exists a boundary strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(52) Let A_0 be a non empty proper subset of X. Suppose A_0 is dense. Then there exists a dense proper strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(53) Let X_1 be a boundary non empty subspace of X. Then there exists a dense proper strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(54) Let X_1 be a dense proper non empty subspace of X. Then there exists a boundary strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(55) Let Y_1, Y_2 be non empty topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is a boundary subspace of X if and only if Y_2 is a boundary subspace of X.

5. Everywhere and Nowhere Dense Subspaces of Non-almost-discrete Spaces

One can prove the following two propositions:

(56) For every non empty topological space X such that every subspace of X is non nowhere dense holds X is almost discrete.

(57) For every non trivial non empty topological space X such that every proper subspace of X is non everywhere dense holds X is almost discrete.

Let X be an almost discrete non empty topological space. One can check the following observations:

* every non empty subspace of X is non nowhere dense,
* every subspace of X which is proper is also non everywhere dense,
* every subspace of X which is everywhere dense is also non proper,
* every non empty subspace of X which is boundary is also non closed,
* every non empty subspace of X which is closed is also non boundary,
* every subspace of X which is dense and proper is also non open,
* every subspace of X which is dense and open is also non proper, and
* every subspace of X which is open and proper is also non dense.

Let X be an almost discrete non empty topological space. One can check that there exists a subspace of X which is non nowhere dense, strict, and non empty.

Let X be an almost discrete non trivial non empty topological space. Observe that there exists a subspace of X which is non everywhere dense and strict.

One can prove the following propositions:

(58) For every non empty topological space X such that there exists a non empty subspace of X which is nowhere dense holds X is non almost discrete.

(59) For every non empty topological space X such that there exists a non empty subspace of X which is boundary and closed holds X is non almost discrete.

(60) For every non empty topological space X such that there exists a non empty subspace of X which is everywhere dense and proper holds X is non almost discrete.

(61) For every non empty topological space X such that there exists a non empty subspace of X which is dense, open, and proper holds X is non almost discrete.

Let X be a non almost discrete non empty topological space. Note that there exists a subspace of X which is nowhere dense, strict, and non empty and there exists a subspace of X which is everywhere dense, proper, strict, and non empty.

In the sequel X is a non almost discrete non empty topological space.

One can prove the following propositions:

(62) Let A_0 be a non empty subset of X. Suppose A_0 is nowhere dense. Then there exists a nowhere dense strict non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(63) Let A_0 be a non empty proper subset of X. Suppose A_0 is everywhere dense. Then there exists an everywhere dense proper strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(64) Let X_1 be a nowhere dense non empty subspace of X. Then there exists an everywhere dense proper strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(65) Let X_1 be an everywhere dense proper non empty subspace of X. Then there exists a nowhere dense strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(66) Let Y_1, Y_2 be non empty topological spaces. Suppose $Y_2 =$ the topological structure of Y_1. Then Y_1 is a nowhere dense subspace of X if and only if Y_2 is a nowhere dense subspace of X.

Let X be a non almost discrete non empty topological space. Observe that there exists a subspace of X which is boundary, closed, strict, and non empty and there exists a subspace of X which is dense, open, proper, strict, and non empty.

The following propositions are true:

(67) Let A_0 be a non empty subset of X. Suppose A_0 is boundary and closed. Then there exists a boundary closed strict non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(68) Let A_0 be a non empty proper subset of X. Suppose A_0 is dense and open. Then there exists a dense open proper strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(69) Let X_1 be a boundary closed non empty subspace of X. Then there exists a dense open proper strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(70) Let X_1 be a dense open proper non empty subspace of X. Then there exists a boundary closed strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition.

(71) Let X_0 be a non empty subspace of X. Then X_0 is nowhere dense if and only if there exists a boundary closed strict non empty subspace X_1 of X such that X_0 is a subspace of X_1.

(72) Let X_0 be a nowhere dense non empty subspace of X. Then

(i) X_0 is boundary and closed, or

(ii) there exists an everywhere dense proper strict non empty subspace X_1 of X and there exists a boundary closed strict non empty subspace X_2 of X such that $X_1 \cap X_2 =$ the topological structure of X_0 and $X_1 \cup X_2 =$ the topological structure of X.

(73) Let X_0 be an everywhere dense non empty subspace of X. Then

(i) X_0 is dense and open, or

(ii) there exists a dense open proper strict non empty subspace X_1 of X and there exists a nowhere dense strict non empty subspace X_2 of X such that X_1 misses X_2 and $X_1 \cup X_2 =$ the topological structure of X_0.

(74) Let X_0 be a nowhere dense non empty subspace of X. Then there exists a dense open proper strict non empty subspace X_1 of X and there exists a boundary closed strict non empty subspace X_2 of X such that X_1 and X_2 constitute a decomposition and X_0 is a subspace of X_2.

(75) Let X_0 be an everywhere dense proper subspace of X. Then there exists a dense open proper strict subspace X_1 of X and there exists a boundary closed strict subspace X_2 of X such that X_1 and X_2 constitute a decomposition and X_1 is a subspace of X_0.

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