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The Lattice of Domains of an Extremally Disconnected Space[1]

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space and let A be a subset of X. Recall that A is said to be a *domain* in X provided $\mathrm{Int}\overline{A} \subseteq A \subseteq \overline{\mathrm{Int}A}$ (see [19], [7]). Recall also that A is said to be a(n) *closed* (*open*) *domain* in X if $A = \overline{\mathrm{Int}A}$ ($A = \mathrm{Int}\overline{A}$, resp.) (see e.g. [9], [19]). It is well-known that for a given topological space all its closed domains form a Boolean lattice, and similarly all its open domains form a Boolean lattice, too (see e.g., [10], [2]). In [17] it is proved that all domains of a given topological space form a complemented lattice. One may ask whether the lattice of all domains is Boolean. The aim is to give an answer to this question.

To present the main results we first recall the definition of a class of topological spaces which is important here. X is called *extremally disconnected* if for every open subset A of X the closure $\overline{A}$ is open in X [13] (comp. [6]). It is shown here, using Mizar System, that *the lattice of all domains of a topological space X is modular iff X is extremally disconnected.* Moreover, for every extremally disconnected space the lattice of all its domains coincides with both the lattice of all its closed domains and the lattice of all its open domains. From these facts it follows that *the lattice of all domains of a topological space X is Boolean iff X is extremally disconnected.*

Note that we also review some of the standard facts on discrete, anti-discrete, almost discrete, extremally disconnected and hereditarily extremally disconnected topological spaces (comp. [9], [6]).

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The articles [14], [5], [16], [11], [18], [12], [19], [3], [15], [4], [20], [1], [17], and [8] provide the notation and terminology for this paper.

1. Selected Properties of Subsets of a Topological Space

In this paper X is a topological space and C is a subset of X.

The following three propositions are true:

(2)[1] $\overline{C} = (\mathrm{Int}(C^{\mathrm{c}}))^{\mathrm{c}}$.

(3) $\overline{C^{\mathrm{c}}} = (\mathrm{Int}C)^{\mathrm{c}}$.

(4) $\mathrm{Int}(C^{\mathrm{c}}) = \overline{C}^{\mathrm{c}}$.

[1] *Editor's Note:* This work has won the 1992 Śleszyński's Award of the Mizar Society.
[1] The proposition (1) has been removed.

In the sequel A, B denote subsets of X.
One can prove the following propositions:

(6)[2] Suppose $A \cup B =$ the carrier of X. Then
(i) if A is closed, then $A \cup \mathrm{Int} B =$ the carrier of X, and
(ii) if B is closed, then $\mathrm{Int} A \cup B =$ the carrier of X.

(7) A is open and closed iff $\overline{A} = \mathrm{Int} A$.

(8) If A is open and closed, then $\mathrm{Int}\overline{A} = \overline{\mathrm{Int} A}$.

(9) If A is condensed and $\overline{\mathrm{Int} A} \subseteq \mathrm{Int}\overline{A}$, then A is open condensed and closed condensed.

(10) If A is condensed and $\overline{\mathrm{Int} A} \subseteq \mathrm{Int}\overline{A}$, then A is open and closed.

(11) If A is condensed, then $\mathrm{Int}\overline{A} = \mathrm{Int} A$ and $\overline{A} = \overline{\mathrm{Int} A}$.

2. Discrete Topological Structures

Let I_1 be a topological structure. We say that I_1 is discrete if and only if:

(Def. 1) The topology of $I_1 = 2^{\text{the carrier of } I_1}$.

We say that I_1 is anti-discrete if and only if:

(Def. 2) The topology of $I_1 = \{\emptyset, \text{the carrier of } I_1\}$.

The following propositions are true:

(12) Let Y be a topological structure. Suppose Y is discrete and anti-discrete. Then $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$.

(13) Let Y be a topological structure. Suppose $\emptyset \in$ the topology of Y and the carrier of $Y \in$ the topology of Y. Suppose $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$. Then Y is discrete and anti-discrete.

Let us observe that there exists a topological structure which is discrete, anti-discrete, strict, and non empty.

The following propositions are true:

(14) Let Y be a discrete topological structure and A be a subset of Y. Then (the carrier of $Y) \setminus A \in$ the topology of Y.

(15) Let Y be an anti-discrete topological structure and A be a subset of Y. Suppose $A \in$ the topology of Y. Then (the carrier of $Y) \setminus A \in$ the topology of Y.

One can check that every topological structure which is discrete is also topological space-like and every topological structure which is anti-discrete is also topological space-like.

One can prove the following proposition

(16) Let Y be a topological space-like topological structure. Suppose $2^{\text{the carrier of } Y} = \{\emptyset, \text{the carrier of } Y\}$. Then Y is discrete and anti-discrete.

Let I_1 be a topological structure. We say that I_1 is almost discrete if and only if:

(Def. 3) For every subset A of I_1 such that $A \in$ the topology of I_1 holds (the carrier of $I_1) \setminus A \in$ the topology of I_1.

Let us mention that every topological structure which is discrete is also almost discrete and every topological structure which is anti-discrete is also almost discrete.

Let us note that there exists a topological structure which is almost discrete and strict.

[2] The proposition (5) has been removed.

3. Discrete Topological Spaces

Let us note that there exists a topological space which is discrete, anti-discrete, strict, and non empty.

The following propositions are true:

(17) X is discrete iff every subset of X is open.

(18) X is discrete iff every subset of X is closed.

(19) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds A is open, then X is discrete.

Let X be a discrete non empty topological space. One can check that every subspace of X is open, closed, and discrete.

Let X be a discrete non empty topological space. Observe that there exists a subspace of X which is discrete and strict.

We now state three propositions:

(20) X is anti-discrete if and only if for every subset A of X such that A is open holds $A = \emptyset$ or $A =$ the carrier of X.

(21) X is anti-discrete if and only if for every subset A of X such that A is closed holds $A = \emptyset$ or $A =$ the carrier of X.

(22) Suppose that for every subset A of X and for every point x of X such that $A = \{x\}$ holds $\overline{A} =$ the carrier of X. Then X is anti-discrete.

Let X be an anti-discrete non empty topological space. One can verify that every subspace of X is anti-discrete.

Let X be an anti-discrete non empty topological space. Note that there exists a subspace of X which is anti-discrete.

Next we state four propositions:

(23) X is almost discrete iff for every subset A of X such that A is open holds A is closed.

(24) X is almost discrete iff for every subset A of X such that A is closed holds A is open.

(25) X is almost discrete iff for every subset A of X such that A is open holds $\overline{A} = A$.

(26) X is almost discrete iff for every subset A of X such that A is closed holds $\operatorname{Int} A = A$.

One can verify that there exists a topological space which is almost discrete and strict.

The following propositions are true:

(27) If for every subset A of X and for every point x of X such that $A = \{x\}$ holds $\overline{A}$ is open, then X is almost discrete.

(28) X is discrete if and only if the following conditions are satisfied:

(i) X is almost discrete, and

(ii) for every subset A of X and for every point x of X such that $A = \{x\}$ holds A is closed.

Let us observe that every topological space which is discrete is also almost discrete and every topological space which is anti-discrete is also almost discrete.

Let X be an almost discrete non empty topological space. Note that every non empty subspace of X is almost discrete.

Let X be an almost discrete non empty topological space. Note that every subspace of X which is open is also closed and every subspace of X which is closed is also open.

Let X be an almost discrete non empty topological space. Note that there exists a subspace of X which is almost discrete, strict, and non empty.

4. Extremally Disconnected Topological Spaces

Let I_1 be a non empty topological space. We say that I_1 is extremally disconnected if and only if:

(Def. 4) For every subset A of I_1 such that A is open holds $\overline{A}$ is open.

Let us note that there exists a non empty topological space which is extremally disconnected and strict.

In the sequel X is a non empty topological space.

We now state a number of propositions:

(29) X is extremally disconnected iff for every subset A of X such that A is closed holds $\mathrm{Int} A$ is closed.

(30) X is extremally disconnected if and only if for all subsets A, B of X such that A is open and B is open holds if A misses B, then $\overline{A}$ misses $\overline{B}$.

(31) X is extremally disconnected if and only if for all subsets A, B of X such that A is closed and B is closed holds if $A \cup B =$ the carrier of X, then $\mathrm{Int} A \cup \mathrm{Int} B =$ the carrier of X.

(32) X is extremally disconnected iff for every subset A of X such that A is open holds $\overline{A} = \mathrm{Int}\overline{A}$.

(33) X is extremally disconnected iff for every subset A of X such that A is closed holds $\mathrm{Int} A = \overline{\mathrm{Int} A}$.

(34) X is extremally disconnected if and only if for every subset A of X such that A is condensed holds A is closed and open.

(35) X is extremally disconnected if and only if for every subset A of X such that A is condensed holds A is closed condensed and open condensed.

(36) X is extremally disconnected iff for every subset A of X such that A is condensed holds $\mathrm{Int}\overline{A} = \overline{\mathrm{Int} A}$.

(37) X is extremally disconnected iff for every subset A of X such that A is condensed holds $\mathrm{Int} A = \overline{A}$.

(38) X is extremally disconnected if and only if for every subset A of X holds if A is open condensed, then A is closed condensed and if A is closed condensed, then A is open condensed.

Let I_1 be a non empty topological space. We say that I_1 is hereditarily extremally disconnected if and only if:

(Def. 5) Every non empty subspace of I_1 is extremally disconnected.

Let us observe that there exists a non empty topological space which is hereditarily extremally disconnected and strict.

Let us mention that every non empty topological space which is hereditarily extremally disconnected is also extremally disconnected and every non empty topological space which is almost discrete is also hereditarily extremally disconnected.

One can prove the following proposition

(39) Let X be an extremally disconnected non empty topological space, X_0 be a non empty subspace of X, and A be a subset of X. Suppose $A =$ the carrier of X_0 and A is dense. Then X_0 is extremally disconnected.

Let X be an extremally disconnected non empty topological space. One can check that every non empty subspace of X which is open is also extremally disconnected.

Let X be an extremally disconnected non empty topological space. Observe that there exists a non empty subspace of X which is extremally disconnected and strict.

Let X be a hereditarily extremally disconnected non empty topological space. One can verify that every non empty subspace of X is hereditarily extremally disconnected.

Let X be a hereditarily extremally disconnected non empty topological space. One can verify that there exists a non empty subspace of X which is hereditarily extremally disconnected and strict.

One can prove the following proposition

(40) If every closed non empty subspace of X is extremally disconnected, then X is hereditarily extremally disconnected.

5. The Lattice of Domains of Extremally Disconnected Spaces

In the sequel Y denotes an extremally disconnected non empty topological space.

Next we state a number of propositions:

(41) The domains of $Y =$ the closed domains of Y.

(42) D-Union$(Y) =$ CLD-Union(Y) and D-Meet$(Y) =$ CLD-Meet(Y).

(43) The lattice of domains of $Y =$ the lattice of closed domains of Y.

(44) The domains of $Y =$ the open domains of Y.

(45) D-Union$(Y) =$ OPD-Union(Y) and D-Meet$(Y) =$ OPD-Meet(Y).

(46) The lattice of domains of $Y =$ the lattice of open domains of Y.

(47) For all elements A, B of the domains of Y holds $(\text{D-Union}(Y))(A, B) = A \cup B$ and $(\text{D-Meet}(Y))(A, B) = A \cap B$.

(48) Let a, b be elements of the lattice of domains of Y and A, B be elements of the domains of Y. If $a = A$ and $b = B$, then $a \sqcup b = A \cup B$ and $a \sqcap b = A \cap B$.

(49) Let F be a family of subsets of Y. Suppose F is domains-family. Let S be a subset of the lattice of domains of Y. If $S = F$, then $\bigsqcup_{(\text{the lattice of domains of } Y)} S = \overline{\bigcup F}$.

(50) Let F be a family of subsets of Y. Suppose F is domains-family. Let S be a subset of the lattice of domains of Y such that $S = F$. Then

(i) if $S \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } Y)} S = \operatorname{Int} \bigcap F$, and

(ii) if $S = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } Y)} S = \Omega_Y$.

In the sequel X is a non empty topological space.

Next we state several propositions:

(51) X is extremally disconnected iff the lattice of domains of X is a modular lattice.

(52) If the lattice of domains of $X =$ the lattice of closed domains of X, then X is extremally disconnected.

(53) If the lattice of domains of $X =$ the lattice of open domains of X, then X is extremally disconnected.

(54) Suppose the lattice of closed domains of $X =$ the lattice of open domains of X. Then X is extremally disconnected.

(55) X is extremally disconnected iff the lattice of domains of X is a Boolean lattice.

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