JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

Totally Bounded Metric Spaces

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MML Identifier: TBSP_1.

WWW: http://mizar.org/JFM/Vol3/tbsp_1.html

The articles [16], [18], [17], [10], [1], [19], [2], [5], [3], [12], [6], [8], [13], [7], [11], [15], [4], [9], and [14] provide the notation and terminology for this paper.

For simplicity, we follow the rules: M denotes a non empty metric space, c denotes an element of M, N denotes a non empty metric structure, w denotes an element of N, G denotes a family of subsets of N, C denotes a subset of N, R denotes a Reflexive non empty metric structure, T denotes a Reflexive symmetric triangle non empty metric structure, t_1 denotes an element of T, Y denotes a family of subsets of T, f denotes a function, n, m, p, k denote natural numbers, r, s, L denote real numbers, and x denotes a set.

Next we state three propositions:

(1) For every L such that $0 < L$ and $L < 1$ and for all n, m such that $n \leq m$ holds $L^m \leq L^n$.

(2) For every L such that $0 < L$ and $L < 1$ and for every k holds $L^k \leq 1$ and $0 < L^k$.

(3) For every L such that $0 < L$ and $L < 1$ and for every s such that $0 < s$ there exists n such that $L^n < s$.

Let us consider N. We say that N is totally bounded if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let given r. Suppose $r > 0$. Then there exists G such that G is finite and the carrier of $N = \bigcup G$ and for every C such that $C \in G$ there exists w such that $C = \mathrm{Ball}(w, r)$.

Let us consider N. We see that the sequence of N is a function and it can be characterized by the following (equivalent) condition:

(Def. 2) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and $\mathrm{rng}\,\mathrm{it} \subseteq$ the carrier of N.

In the sequel S_1 is a sequence of M and S_2 is a sequence of N.

We now state the proposition

(5)[1] f is a sequence of N iff $\mathrm{dom}\,f = \mathbb{N}$ and for every n holds $f(n)$ is an element of N.

Let us consider N, S_2. We say that S_2 is convergent if and only if:

(Def. 3) There exists an element x of N such that for every r such that $r > 0$ there exists n such that for every m such that $n \leq m$ holds $\rho(S_2(m), x) < r$.

Let us consider M, S_1. Let us assume that S_1 is convergent. The functor $\lim S_1$ yields an element of M and is defined by:

[1] The proposition (4) has been removed.

(Def. 4) For every r such that $r > 0$ there exists n such that for every m such that $m \geq n$ holds $\rho(S_1(m), \lim S_1) < r$.

Let us consider N, S_2. We say that S_2 is Cauchy if and only if:

(Def. 5) For every r such that $r > 0$ there exists p such that for all n, m such that $p \leq n$ and $p \leq m$ holds $\rho(S_2(n), S_2(m)) < r$.

Let us consider N. We say that N is complete if and only if:

(Def. 6) For every S_2 such that S_2 is Cauchy holds S_2 is convergent.

Next we state the proposition

(7)[2] If N is triangle and symmetric and S_2 is convergent, then S_2 is Cauchy.

Let M be a triangle symmetric non empty metric structure. Note that every sequence of M which is convergent is also Cauchy.

One can prove the following propositions:

(8) Suppose N is symmetric. Then S_2 is Cauchy if and only if for every r such that $r > 0$ there exists p such that for all n, k such that $p \leq n$ holds $\rho(S_2(n+k), S_2(n)) < r$.

(9) Let f be a contraction of M. Suppose M is complete. Then there exists c such that $f(c) = c$ and for every element y of M such that $f(y) = y$ holds $y = c$.

(10) If $T_{\rm top}$ is compact, then T is complete.

(12)[3] If N is Reflexive and triangle and $N_{\rm top}$ is compact, then N is totally bounded.

Let us consider N. We say that N is bounded if and only if:

(Def. 8)[4] There exists r such that $0 < r$ and for all points x, y of N holds $\rho(x, y) \leq r$.

Let C be a subset of N. We say that C is bounded if and only if:

(Def. 9) There exists r such that $0 < r$ and for all points x, y of N such that $x \in C$ and $y \in C$ holds $\rho(x, y) \leq r$.

Let A be a non empty set. Observe that the discrete space on A is bounded.

One can check that there exists a non empty metric space which is bounded.

We now state several propositions:

(14)[5] $\emptyset_N$ is bounded.

(15) Let C be a subset of N. Then

(i) if $C \neq \emptyset$ and C is bounded, then there exist r, w such that $0 < r$ and $w \in C$ and for every point z of N such that $z \in C$ holds $\rho(w, z) \leq r$, and

(ii) if N is symmetric and triangle and there exist r, w such that $0 < r$ and $w \in C$ and for every point z of N such that $z \in C$ holds $\rho(w, z) \leq r$, then C is bounded.

(16) If N is Reflexive and $0 < r$, then $w \in \mathrm{Ball}(w, r)$ and $\mathrm{Ball}(w, r) \neq \emptyset$.

(17) If $r \leq 0$, then $\mathrm{Ball}(t_1, r) = \emptyset$.

(19)[6] $\mathrm{Ball}(t_1, r)$ is bounded.

(20) For all subsets P, Q of T such that P is bounded and Q is bounded holds $P \cup Q$ is bounded.

[2] The proposition (6) has been removed.
[3] The proposition (11) has been removed.
[4] The definition (Def. 7) has been removed.
[5] The proposition (13) has been removed.
[6] The proposition (18) has been removed.

(21) For all subsets C, D of N such that C is bounded and $D \subseteq C$ holds D is bounded.

(22) For every subset P of T such that $P = \{t_1\}$ holds P is bounded.

(23) For every subset P of T such that P is finite holds P is bounded.

Let us consider T. One can verify that every subset of T which is finite is also bounded.
Let us consider T. Observe that there exists a subset of T which is finite and non empty.
We now state two propositions:

(24) If Y is finite and for every subset P of T such that $P \in Y$ holds P is bounded, then $\bigcup Y$ is bounded.

(25) N is bounded iff Ω_N is bounded.

Let N be a bounded non empty metric structure. Note that Ω_N is bounded.
Next we state the proposition

(26) If T is totally bounded, then T is bounded.

Let N be a Reflexive non empty metric structure and let C be a subset of N. Let us assume that C is bounded. The functor $\varnothing C$ yielding a real number is defined by:

(Def. 10)(i) For all points x, y of N such that $x \in C$ and $y \in C$ holds $\rho(x,y) \leq \varnothing C$ and for every s such that for all points x, y of N such that $x \in C$ and $y \in C$ holds $\rho(x,y) \leq s$ holds $\varnothing C \leq s$ if $C \neq \emptyset$,

(ii) $\varnothing C = 0$, otherwise.

The following propositions are true:

(28)[7] For every subset P of T such that $P = \{x\}$ holds $\varnothing P = 0$.

(29) For every subset S of R such that S is bounded holds $0 \leq \varnothing S$.

(30) For every subset A of M such that $A \neq \emptyset$ and A is bounded and $\varnothing A = 0$ there exists a point g of M such that $A = \{g\}$.

(31) If $0 < r$, then $\varnothing \operatorname{Ball}(t_1, r) \leq 2 \cdot r$.

(32) For all subsets S, V of R such that S is bounded and $V \subseteq S$ holds $\varnothing V \leq \varnothing S$.

(33) For all subsets P, Q of T such that P is bounded and Q is bounded and P meets Q holds $\varnothing(P \cup Q) \leq \varnothing P + \varnothing Q$.

Let us consider N, S_2. Then $\operatorname{rng} S_2$ is a subset of N.
Next we state the proposition

(34) For every sequence S_1 of T such that S_1 is Cauchy holds $\operatorname{rng} S_1$ is bounded.

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Received September 30, 1991

Published January 2, 2004