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# Some Properties of Binary Relations

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**Summary.** The article contains some theorems on binary relations, which are used in papers [2], [3], [4], and other.

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The articles [5], [1], and [6] provide the notation and terminology for this paper.

We adopt the following convention: $x$, $y$, $X$, $Y$, $Z$, $W$ are sets and $R$, $S$, $T$ are binary relations.

Let us consider $X$, $Y$. One can check that $[:X, Y:]$ is relation-like.

We now state a number of propositions:

(12)[1] If $X \neq \emptyset$ and $Y \neq \emptyset$, then $\mathrm{dom}[:X, Y:] = X$ and $\mathrm{rng}[:X, Y:] = Y$.

(13) $\mathrm{dom}(R \cap [:X, Y:]) \subseteq X$ and $\mathrm{rng}(R \cap [:X, Y:]) \subseteq Y$.

(14) If $X$ misses $Y$, then $\mathrm{dom}(R \cap [:X, Y:])$ misses $\mathrm{rng}(R \cap [:X, Y:])$ and $\mathrm{dom}(R^{\smile} \cap [:X, Y:])$ misses $\mathrm{rng}(R^{\smile} \cap [:X, Y:])$.

(15) If $R \subseteq [:X, Y:]$, then $\mathrm{dom} R \subseteq X$ and $\mathrm{rng} R \subseteq Y$.

(16) If $R \subseteq [:X, Y:]$, then $R^{\smile} \subseteq [:Y, X:]$.

(18)[2] $[:X, Y:]^{\smile} = [:Y, X:]$.

(20)[3] $(R \cup S) \cdot T = R \cdot T \cup S \cdot T$ and $R \cdot (S \cup T) = R \cdot S \cup R \cdot T$.

(22)[4](i) If $X$ misses $Y$ and $R \subseteq [:X, Y:] \cup [:Y, X:]$ and $\langle x, y \rangle \in R$ and $x \in X$, then $x \notin Y$ and $y \notin X$ and $y \in Y$,

(ii) if $X$ misses $Y$ and $R \subseteq [:X, Y:] \cup [:Y, X:]$ and $\langle x, y \rangle \in R$ and $y \in Y$, then $y \notin X$ and $x \notin Y$ and $x \in X$,

(iii) if $X$ misses $Y$ and $R \subseteq [:X, Y:] \cup [:Y, X:]$ and $\langle x, y \rangle \in R$ and $x \in Y$, then $x \notin X$ and $y \notin Y$ and $y \in X$, and

(iv) if $X$ misses $Y$ and $R \subseteq [:X, Y:] \cup [:Y, X:]$ and $\langle x, y \rangle \in R$ and $y \in X$, then $x \notin X$ and $y \notin Y$ and $x \in Y$.

(24)[5] If $R \subseteq [:X, Y:]$ and $Z \subseteq X$, then $R \restriction Z = R \cap [:Z, Y:]$ and if $R \subseteq [:X, Y:]$ and $Z \subseteq Y$, then $Z \restriction R = R \cap [:X, Z:]$.

[1] The propositions (1)–(11) have been removed.
[2] The proposition (17) has been removed.
[3] The proposition (19) has been removed.
[4] The proposition (21) has been removed.
[5] The proposition (23) has been removed.

(25) If $R \subseteq [:X, Y:]$ and $X = Z \cup W$, then $R = R \restriction Z \cup R \restriction W$.

(26) If $X$ misses $Y$ and $R \subseteq [:X, Y:] \cup [:Y, X:]$, then $R \restriction X \subseteq [:X, Y:]$.

(27) If $R \subseteq S$, then $R^\smile \subseteq S^\smile$.

(29)[6] $\mathrm{id}_X \cdot \mathrm{id}_X = \mathrm{id}_X$.

(30) $\mathrm{id}_{\{x\}} = \{\langle x, x\rangle\}$.

(31) $\langle x, y\rangle \in \mathrm{id}_X$ iff $\langle y, x\rangle \in \mathrm{id}_X$.

(32) $\mathrm{id}_{X \cup Y} = \mathrm{id}_X \cup \mathrm{id}_Y$ and $\mathrm{id}_{X \cap Y} = \mathrm{id}_X \cap \mathrm{id}_Y$ and $\mathrm{id}_{X \setminus Y} = \mathrm{id}_X \setminus \mathrm{id}_Y$.

(33) If $X \subseteq Y$, then $\mathrm{id}_X \subseteq \mathrm{id}_Y$.

(34) $\mathrm{id}_{X \setminus Y} \setminus \mathrm{id}_X = \emptyset$.

(35) If $R \subseteq \mathrm{id}_{\mathrm{dom}\, R}$, then $R = \mathrm{id}_{\mathrm{dom}\, R}$.

(36) If $\mathrm{id}_X \subseteq R \cup R^\smile$, then $\mathrm{id}_X \subseteq R$ and $\mathrm{id}_X \subseteq R^\smile$.

(37) If $\mathrm{id}_X \subseteq R$, then $\mathrm{id}_X \subseteq R^\smile$.

(38) If $R \subseteq [:X, X:]$, then $R \setminus \mathrm{id}_{\mathrm{dom}\, R} = R \setminus \mathrm{id}_X$ and $R \setminus \mathrm{id}_{\mathrm{rng}\, R} = R \setminus \mathrm{id}_X$.

(39) If $\mathrm{id}_X \cdot (R \setminus \mathrm{id}_X) = \emptyset$, then $\mathrm{dom}(R \setminus \mathrm{id}_X) = \mathrm{dom}\, R \setminus \mathrm{dom}(\mathrm{id}_X)$ and if $(R \setminus \mathrm{id}_X) \cdot \mathrm{id}_X = \emptyset$, then $\mathrm{rng}(R \setminus \mathrm{id}_X) = \mathrm{rng}\, R \setminus \mathrm{rng}(\mathrm{id}_X)$.

(40) If $R \subseteq R \cdot R$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{rng}\, R}) = \emptyset$, then $\mathrm{id}_{\mathrm{rng}\, R} \subseteq R$ and if $R \subseteq R \cdot R$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$, then $\mathrm{id}_{\mathrm{dom}\, R} \subseteq R$.

(41) If $R \subseteq R \cdot R$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{rng}\, R}) = \emptyset$, then $R \cap \mathrm{id}_{\mathrm{rng}\, R} = \mathrm{id}_{\mathrm{rng}\, R}$ and if $R \subseteq R \cdot R$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$, then $R \cap \mathrm{id}_{\mathrm{dom}\, R} = \mathrm{id}_{\mathrm{dom}\, R}$.

(42) If $R \cdot (R \setminus \mathrm{id}_X) = \emptyset$ and $\mathrm{rng}\, R \subseteq X$, then $R \cdot (R \setminus \mathrm{id}_{\mathrm{rng}\, R}) = \emptyset$ and if $(R \setminus \mathrm{id}_X) \cdot R = \emptyset$ and $\mathrm{dom}\, R \subseteq X$, then $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$.

Let us consider $R$. The functor $\mathrm{CL}(R)$ yielding a binary relation is defined as follows:

(Def. 1) $\mathrm{CL}(R) = R \cap \mathrm{id}_{\mathrm{dom}\, R}$.

One can prove the following propositions:

(43) $\mathrm{CL}(R) \subseteq R$ and $\mathrm{CL}(R) \subseteq \mathrm{id}_{\mathrm{dom}\, R}$.

(44) If $\langle x, y\rangle \in \mathrm{CL}(R)$, then $x \in \mathrm{dom}\, \mathrm{CL}(R)$ and $x = y$.

(45) $\mathrm{dom}\, \mathrm{CL}(R) = \mathrm{rng}\, \mathrm{CL}(R)$.

(46)(i) $x \in \mathrm{dom}\, \mathrm{CL}(R)$ iff $x \in \mathrm{dom}\, R$ and $\langle x, x\rangle \in R$,

(ii) $x \in \mathrm{rng}\, \mathrm{CL}(R)$ iff $x \in \mathrm{dom}\, R$ and $\langle x, x\rangle \in R$,

(iii) $x \in \mathrm{rng}\, \mathrm{CL}(R)$ iff $x \in \mathrm{rng}\, R$ and $\langle x, x\rangle \in R$, and

(iv) $x \in \mathrm{dom}\, \mathrm{CL}(R)$ iff $x \in \mathrm{rng}\, R$ and $\langle x, x\rangle \in R$.

(47) $\mathrm{CL}(R) = \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}$.

(48)(i) If $R \cdot R = R$ and $R \cdot (R \setminus \mathrm{CL}(R)) = \emptyset$ and $\langle x, y\rangle \in R$ and $x \neq y$, then $x \in \mathrm{dom}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $y \in \mathrm{dom}\, \mathrm{CL}(R)$, and

(ii) if $R \cdot R = R$ and $(R \setminus \mathrm{CL}(R)) \cdot R = \emptyset$ and $\langle x, y\rangle \in R$ and $x \neq y$, then $y \in \mathrm{rng}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $x \in \mathrm{dom}\, \mathrm{CL}(R)$.

[6] The proposition (28) has been removed.

(49)(i) If $R \cdot R = R$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$ and $\langle x, y \rangle \in R$ and $x \neq y$, then $x \in \mathrm{dom}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $y \in \mathrm{dom}\, \mathrm{CL}(R)$, and

(ii) if $R \cdot R = R$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$ and $\langle x, y \rangle \in R$ and $x \neq y$, then $y \in \mathrm{rng}\, R \setminus \mathrm{dom}\, \mathrm{CL}(R)$ and $x \in \mathrm{dom}\, \mathrm{CL}(R)$.

(50) If $R \cdot R = R$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$, then $\mathrm{dom}\, \mathrm{CL}(R) = \mathrm{rng}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) = \mathrm{rng}\, R$ and if $R \cdot R = R$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$, then $\mathrm{dom}\, \mathrm{CL}(R) = \mathrm{dom}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) = \mathrm{dom}\, R$.

(51) $\mathrm{dom}\, \mathrm{CL}(R) \subseteq \mathrm{dom}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) \subseteq \mathrm{rng}\, R$ and $\mathrm{rng}\, \mathrm{CL}(R) \subseteq \mathrm{dom}\, R$ and $\mathrm{dom}\, \mathrm{CL}(R) \subseteq \mathrm{rng}\, R$.

(52) $\mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)} \subseteq \mathrm{id}_{\mathrm{dom}\, R}$ and $\mathrm{id}_{\mathrm{rng}\, \mathrm{CL}(R)} \subseteq \mathrm{id}_{\mathrm{dom}\, R}$.

(53) $\mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)} \subseteq R$ and $\mathrm{id}_{\mathrm{rng}\, \mathrm{CL}(R)} \subseteq R$.

(54) If $\mathrm{id}_X \subseteq R$ and $\mathrm{id}_X \cdot (R \setminus \mathrm{id}_X) = \emptyset$, then $R \restriction X = \mathrm{id}_X$ and if $\mathrm{id}_X \subseteq R$ and $(R \setminus \mathrm{id}_X) \cdot \mathrm{id}_X = \emptyset$, then $X \restriction R = \mathrm{id}_X$.

(55)(i) If $\mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)} \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}) = \emptyset$, then $R \restriction \mathrm{dom}\, \mathrm{CL}(R) = \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}$ and $R \restriction \mathrm{rng}\, \mathrm{CL}(R) = \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}$, and

(ii) if $(R \setminus \mathrm{id}_{\mathrm{rng}\, \mathrm{CL}(R)}) \cdot \mathrm{id}_{\mathrm{rng}\, \mathrm{CL}(R)} = \emptyset$, then $\mathrm{dom}\, \mathrm{CL}(R) \restriction R = \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}$ and $\mathrm{rng}\, \mathrm{CL}(R) \restriction R = \mathrm{id}_{\mathrm{rng}\, \mathrm{CL}(R)}$.

(56) If $R \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$, then $\mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)} \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}) = \emptyset$ and if $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$, then $(R \setminus \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)}) \cdot \mathrm{id}_{\mathrm{dom}\, \mathrm{CL}(R)} = \emptyset$.

(57) If $S \cdot R = S$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$, then $S \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$ and if $R \cdot S = S$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$, then $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot S = \emptyset$.

(58) If $S \cdot R = S$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$, then $\mathrm{CL}(S) \subseteq \mathrm{CL}(R)$ and if $R \cdot S = S$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$, then $\mathrm{CL}(S) \subseteq \mathrm{CL}(R)$.

(59)(i) If $S \cdot R = S$ and $R \cdot (R \setminus \mathrm{id}_{\mathrm{dom}\, R}) = \emptyset$ and $R \cdot S = R$ and $S \cdot (S \setminus \mathrm{id}_{\mathrm{dom}\, S}) = \emptyset$, then $\mathrm{CL}(S) = \mathrm{CL}(R)$, and

(ii) if $R \cdot S = S$ and $(R \setminus \mathrm{id}_{\mathrm{dom}\, R}) \cdot R = \emptyset$ and $S \cdot R = R$ and $(S \setminus \mathrm{id}_{\mathrm{dom}\, S}) \cdot S = \emptyset$, then $\mathrm{CL}(S) = \mathrm{CL}(R)$.

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