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Construction of a bilinear antisymmetric form in symplectic vector space[1]

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Summary. In this text we will present unpublished results by Eugeniusz Kusak. It contains an axiomatic description of the class of all spaces $\langle V;\ \perp_\xi \rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear antisymmetric form i.e. $\xi(x,y) = -\xi(y,x)$ and $x \perp_\xi y$ iff $\xi(x,y) = 0$ for $x,\ y \in V$. It also contains an effective construction of bilinear antisymmetric form ξ for given symplectic space $\langle V;\ \perp \rangle$ such that $\perp = \perp_\xi$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a,b,x)$ for $a,b,x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y + \varepsilon z \,\&\, y \perp z + \varepsilon x \Rightarrow z \perp x + \varepsilon y$. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry - see [5].

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The articles [6], [3], [8], [1], [2], [7], [4], and [9] provide the notation and terminology for this paper.

In this paper F denotes a field.

Let us consider F. We consider simplectic structures over F as extensions of vector space structure over F as systems

⟨ a carrier, an addition, a zero, a left multiplication, an orthogonality ⟩,

where the carrier is a set, the addition is a binary operation on the carrier, the zero is an element of the carrier, the left multiplication is a function from [: the carrier of F, the carrier :] into the carrier, and the orthogonality is a binary relation on the carrier.

Let us consider F. Note that there exists a simplectic structure over F which is non empty.

Let us consider F, let S be a simplectic structure over F, and let a, b be elements of S. The predicate $a \perp b$ is defined as follows:

(Def. 1) $\langle a, b \rangle \in$ the orthogonality of S.

Let us consider F, let X be a non empty set, let m_1 be a binary operation on X, let o be an element of X, let m_2 be a function from [: the carrier of F, X :] into X, and let m_3 be a binary relation on X. Observe that $\langle X, m_1, o, m_2, m_3 \rangle$ is non empty.

Let us consider F. One can verify that there exists a non empty simplectic structure over F which is Abelian, add-associative, right zeroed, and right complementable.

Let us consider F and let I_1 be an Abelian add-associative right zeroed right complementable non empty simplectic structure over F. We say that I_1 is simplectic space-like if and only if the condition (Def. 2) is satisfied.

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(Def. 2) Let a, b, c, x be elements of I_1 and l be an element of F. Then

(i) if $a \neq 0_{(I_1)}$, then there exists an element y of the carrier of I_1 such that $y \not\perp a$,

(ii) if $a \perp b$, then $l \cdot a \perp b$,

(iii) if $b \perp a$ and $c \perp a$, then $b + c \perp a$,

(iv) if $b \not\perp a$, then there exists an element k of F such that $x - k \cdot b \perp a$, and

(v) if $a \perp b + c$ and $b \perp c + a$, then $c \perp a + b$.

Let us consider F. Note that there exists an Abelian add-associative right zeroed right complementable non empty simplectic structure over F which is simplectic space-like, vector space-like, and strict.

Let us consider F. A simplectic space over F is a simplectic space-like vector space-like Abelian add-associative right zeroed right complementable non empty simplectic structure over F.

We adopt the following rules: S denotes a simplectic space over F, a, b, c, d, a', b', p, q, x, y, z denote elements of S, and k, l denote elements of F.

One can prove the following propositions:

(11)[1] $0_S \perp a$.

(12) If $a \perp b$, then $b \perp a$.

(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.

(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.

(15) If $b \not\perp a$ and $l \neq 0_F$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.

(16) If $a \perp b$, then $-a \perp b$.

(19)[2] If $a \not\perp c$, then $a + b \not\perp c$ or $(\mathbf{1}_F + \mathbf{1}_F) \cdot a + b \not\perp c$.

(20) If $a' \not\perp a$ and $a' \perp b$ and $b' \not\perp b$ and $b' \perp a$, then $a' + b' \not\perp a$ and $a' + b' \not\perp b$.

(21) If $a \neq 0_S$ and $b \neq 0_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$.

(22) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $a \neq 0_S$ and $b \neq 0_S$ and $c \neq 0_S$, then there exists p such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$.

(23) If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.

(24) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.

(25) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$, then $a \perp a$.

Let us consider F and let us consider S, a, b, x. Let us assume that $b \not\perp a$. The functor $\mathrm{J}(a,b,x)$ yielding an element of F is defined by:

(Def. 6)[3] For every element l of F such that $x - l \cdot b \perp a$ holds $\mathrm{J}(a,b,x) = l$.

The following propositions are true:

(27)[4] If $b \not\perp a$, then $x - \mathrm{J}(a,b,x) \cdot b \perp a$.

(28) If $b \not\perp a$, then $\mathrm{J}(a,b,l \cdot x) = l \cdot \mathrm{J}(a,b,x)$.

(29) If $b \not\perp a$, then $\mathrm{J}(a,b,x+y) = \mathrm{J}(a,b,x) + \mathrm{J}(a,b,y)$.

(30) If $b \not\perp a$ and $l \neq 0_F$, then $\mathrm{J}(a,l \cdot b,x) = l^{-1} \cdot \mathrm{J}(a,b,x)$.

[1] The propositions (1)–(10) have been removed.
[2] The propositions (17) and (18) have been removed.
[3] The definitions (Def. 3)–(Def. 5) have been removed.
[4] The proposition (26) has been removed.

(31) If $b \not\perp a$ and $l \neq 0_F$, then $\mathrm{J}(l \cdot a, b, x) = \mathrm{J}(a, b, x)$.

(32) If $b \not\perp a$ and $p \perp a$, then $\mathrm{J}(a, b+p, c) = \mathrm{J}(a, b, c)$ and $\mathrm{J}(a, b, c+p) = \mathrm{J}(a, b, c)$.

(33) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $\mathrm{J}(a+p, b, c) = \mathrm{J}(a, b, c)$.

(34) If $b \not\perp a$ and $c - b \perp a$, then $\mathrm{J}(a, b, c) = \mathbf{1}_F$.

(35) If $b \not\perp a$, then $\mathrm{J}(a, b, b) = \mathbf{1}_F$.

(36) If $b \not\perp a$, then $x \perp a$ iff $\mathrm{J}(a, b, x) = 0_F$.

(37) If $b \not\perp a$ and $q \not\perp a$, then $\mathrm{J}(a, b, p) \cdot \mathrm{J}(a, b, q)^{-1} = \mathrm{J}(a, q, p)$.

(38) If $b \not\perp a$ and $c \not\perp a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(a, c, b)^{-1}$.

(39) If $b \not\perp a$ and $b \perp c + a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(c, b, a)$.

(40) If $a \not\perp b$ and $c \not\perp b$, then $\mathrm{J}(c, b, a) = (-\mathrm{J}(b, a, c)^{-1}) \cdot \mathrm{J}(a, b, c)$.

(41) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $a \not\perp p$ and $a \not\perp q$ and $b \not\perp p$ and $b \not\perp q$, then $\mathrm{J}(a, p, q) \cdot \mathrm{J}(b, q, p) = \mathrm{J}(p, a, b) \cdot \mathrm{J}(q, b, a)$.

(42) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $\mathrm{J}(a, q, p) \cdot \mathrm{J}(p, a, x) = \mathrm{J}(x, q, p) \cdot \mathrm{J}(q, a, x)$.

(43) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$, then $\mathrm{J}(a, b, p) \cdot \mathrm{J}(p, a, x) \cdot \mathrm{J}(x, p, y) = \mathrm{J}(a, b, q) \cdot \mathrm{J}(q, a, x) \cdot \mathrm{J}(x, q, y)$.

(44) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $\mathrm{J}(p, a, x) \cdot \mathrm{J}(x, p, y) = (-\mathrm{J}(p, a, y)) \cdot \mathrm{J}(y, p, x)$.

Let us consider F, S, x, y, a, b. Let us assume that $b \not\perp a$ and $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$. The functor $x \cdot_{a,b} y$ yielding an element of F is defined by:

(Def. 7)(i) For every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = \mathrm{J}(a, b, q) \cdot \mathrm{J}(q, a, x) \cdot \mathrm{J}(x, q, y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$,

(ii) $x \cdot_{a,b} y = 0_F$ if for every p holds $p \perp a$ or $p \perp x$.

We now state several propositions:

(47)[5] If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $b \not\perp a$ and $x = 0_S$, then $x \cdot_{a,b} y = 0_F$.

(48) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $b \not\perp a$, then $x \cdot_{a,b} y = 0_F$ iff $y \perp x$.

(49) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $b \not\perp a$, then $x \cdot_{a,b} y = -y \cdot_{a,b} x$.

(50) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(51) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and $b \not\perp a$, then $x \cdot_{a,b} (y + z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

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