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Lattice of Substitutions

Adam Grabowski
Warsaw University
Białystok

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The articles [5], [7], [4], [6], [1], [8], [2], [3], [10], and [9] provide the notation and terminology for this paper.

1. Preliminaries

In this paper V, C are sets.

Let us consider V, C. The functor SubstitutionSet(V,C) yields a subset of $\mathrm{Fin}(V \dot{\to} C)$ and is defined by the condition (Def. 1).

(Def. 1) $\mathrm{SubstitutionSet}(V,C) = \{A; A$ ranges over elements of $\mathrm{Fin}(V \dot{\to} C) : \bigwedge_{u\,:\,\mathrm{set}} (u \in A \Rightarrow u$is finite$) \wedge \bigwedge_{s,t\,:\,\text{element of } V \dot{\to} C} (s \in A \wedge t \in A \wedge s \subseteq t \Rightarrow s = t)\}$.

Next we state two propositions:

(1) $\emptyset \in \mathrm{SubstitutionSet}(V,C)$.

(2) $\{\emptyset\} \in \mathrm{SubstitutionSet}(V,C)$.

Let us consider V, C. Observe that SubstitutionSet(V,C) is non empty.

Let us consider V, C and let A, B be elements of SubstitutionSet(V,C). Then $A \cup B$ is an element of $\mathrm{Fin}(V \dot{\to} C)$.

Let us consider V, C. One can check that there exists an element of SubstitutionSet(V,C) which is non empty.

Let us consider V, C. One can verify that every element of SubstitutionSet(V,C) is finite.

Let us consider V, C and let A be an element of $\mathrm{Fin}(V \dot{\to} C)$. The functor μA yielding an element of SubstitutionSet(V,C) is defined by:

(Def. 2) $\mu A = \{t; t$ ranges over elements of $V \dot{\to} C : t$ is finite $\wedge \bigwedge_{s\,:\,\text{element of } V \dot{\to} C} (s \in A \wedge s \subseteq t \Leftrightarrow s = t)\}$.

Let us consider V, C and let A be a non empty element of SubstitutionSet(V,C). Observe that every element of A is function-like and relation-like.

Let us consider V, C. One can verify that every element of $V \dot{\to} C$ is function-like and relation-like.

Let us consider V, C and let A, B be elements of $\mathrm{Fin}(V \dot{\to} C)$. The functor $A \frown B$ yields an element of $\mathrm{Fin}(V \dot{\to} C)$ and is defined by:

(Def. 3) $A \frown B = \{s \cup t; s$ ranges over elements of $V \dot{\to} C, t$ ranges over elements of $V \dot{\to} C : s \in A \wedge t \in B \wedge s \approx t\}$.

In the sequel A, B, D denote elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$.
Next we state four propositions:

(3) $A \frown B = B \frown A$.

(4) If $B = \{\emptyset\}$, then $A \frown B = A$.

(5) For all sets a, b such that $B \in \mathrm{SubstitutionSet}(V, C)$ and $a \in B$ and $b \in B$ and $a \subseteq b$ holds $a = b$.

(6) For every set a such that $a \in \mu B$ holds $a \in B$ and for every set b such that $b \in B$ and $b \subseteq a$ holds $b = a$.

Let us consider V, C. One can check that there exists an element of $V \dot{\rightarrow} C$ which is finite.
Next we state a number of propositions:

(7) For every finite set a such that $a \in B$ and for every finite set b such that $b \in B$ and $b \subseteq a$ holds $b = a$ holds $a \in \mu B$.

(8) $\mu A \subseteq A$.

(9) If $A = \emptyset$, then $\mu A = \emptyset$.

(10) For every finite set b such that $b \in B$ there exists a set c such that $c \subseteq b$ and $c \in \mu B$.

(11) For every element K of $\mathrm{SubstitutionSet}(V, C)$ holds $\mu K = K$.

(12) $\mu(A \cup B) \subseteq \mu A \cup B$.

(13) $\mu(\mu A \cup B) = \mu(A \cup B)$.

(14) If $A \subseteq B$, then $A \frown D \subseteq B \frown D$.

(15) For every set a such that $a \in A \frown B$ there exist sets b, c such that $b \in A$ and $c \in B$ and $a = b \cup c$.

(16) For all elements b, c of $V \dot{\rightarrow} C$ such that $b \in A$ and $c \in B$ and $b \approx c$ holds $b \cup c \in A \frown B$.

(17) $\mu(A \frown B) \subseteq \mu A \frown B$.

(18) If $A \subseteq B$, then $D \frown A \subseteq D \frown B$.

(19) $\mu(\mu A \frown B) = \mu(A \frown B)$.

(20) $\mu(A \frown \mu B) = \mu(A \frown B)$.

(21) For all elements K, L, M of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $K \frown (L \frown M) = (K \frown L) \frown M$.

(22) For all elements K, L, M of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $K \frown (L \cup M) = K \frown L \cup K \frown M$.

(23) $B \subseteq B \frown B$.

(24) $\mu(A \frown A) = \mu A$.

(25) For every element K of $\mathrm{SubstitutionSet}(V, C)$ holds $\mu(K \frown K) = K$.

2. Definition of the lattice

Let us consider V, C. The functor $\mathrm{SubstLatt}(V,C)$ yields a strict lattice structure and is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of $\mathrm{SubstLatt}(V,C) = \mathrm{SubstitutionSet}(V,C)$, and

(ii) for all elements A, B of $\mathrm{SubstitutionSet}(V,C)$ holds (the join operation of $\mathrm{SubstLatt}(V,C))(A, B) = \mu(A \cup B)$ and (the meet operation of $\mathrm{SubstLatt}(V,C))(A, B) = \mu(A \frown B)$.

Let us consider V, C. Observe that $\mathrm{SubstLatt}(V,C)$ is non empty.
Let us consider V, C. Observe that $\mathrm{SubstLatt}(V,C)$ is lattice-like.
Let us consider V, C. Note that $\mathrm{SubstLatt}(V,C)$ is distributive and bounded.
The following propositions are true:

(26) $\bot_{\mathrm{SubstLatt}(V,C)} = \emptyset$.

(27) $\top_{\mathrm{SubstLatt}(V,C)} = \{\emptyset\}$.

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