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Submetric Spaces — Part I[1]

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Summary. Definitions of pseudometric space, nonsymmetric metric space, semimetric space and ultrametric space are introduced. We find some relations between these spaces and prove that every ultrametric space is a metric space. We define the relation *is between*. Moreover we introduce the notions of the open segment and the closed segment.

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The articles [7], [4], [10], [8], [11], [2], [3], [1], [6], [9], and [5] provide the notation and terminology for this paper.

The following propositions are true:

(1) For all elements x, y of $\mathbb{R}$ such that $0 \leq x$ and $0 \leq y$ holds $\max(x,y) \leq x+y$.

(2) For every metric space M and for all elements x, y of M such that $x \neq y$ holds $0 < \rho(x,y)$.

(4)[1] For all elements x, y of $\{\emptyset\}$ such that $x = y$ holds $(\{[\emptyset,\emptyset]\} \longmapsto 0)(x, y) = 0$.

(5) For all elements x, y of $\{\emptyset\}$ such that $x \neq y$ holds $0 < (\{[\emptyset,\emptyset]\} \longmapsto 0)(x, y)$.

(6) For all elements x, y of $\{\emptyset\}$ holds $(\{[\emptyset,\emptyset]\} \longmapsto 0)(x, y) = (\{[\emptyset,\emptyset]\} \longmapsto 0)(y, x)$.

(7) For all elements x, y, z of $\{\emptyset\}$ holds $(\{[\emptyset,\emptyset]\} \longmapsto 0)(x, z) \leq (\{[\emptyset,\emptyset]\} \longmapsto 0)(x, y) + (\{[\emptyset,\emptyset]\} \longmapsto 0)(y, z)$.

(8) For all elements x, y, z of $\{\emptyset\}$ holds $(\{[\emptyset,\emptyset]\} \longmapsto 0)(x, z) \leq \max((\{[\emptyset,\emptyset]\} \longmapsto 0)(x, y), (\{[\emptyset,\emptyset]\} \longmapsto 0)(y, z))$.

A pseudo metric space is a Reflexive symmetric triangle non empty metric structure.

Let A be a non empty set and let f be a function from $[:A, A:]$ into $\mathbb{R}$. We say that f is Discerning if and only if:

(Def. 1) For all elements a, b of A such that $a \neq b$ holds $0 < f(a, b)$.

Let M be a non empty metric structure. We say that M is Discerning if and only if:

(Def. 2) The distance of M is Discerning.

Next we state the proposition

(14)[2] Let M be a non empty metric structure. Then for all elements a, b of M such that $a \neq b$ holds $0 < \rho(a,b)$ if and only if M is Discerning.

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[1]Supported by RPBP.III-24.B3.

[1] The proposition (3) has been removed.

[2] The propositions (9)–(13) have been removed.

Let us mention that $\langle\{\emptyset\},\{[\emptyset,\emptyset]\}\mapsto 0\rangle$ is Reflexive, symmetric, Discerning, and triangle.

Let us observe that there exists a non empty metric structure which is Reflexive, Discerning, symmetric, and triangle.

A semi metric space is a Reflexive Discerning symmetric non empty metric structure.

The following two propositions are true:

(16)[3] For every Discerning non empty metric structure M and for all elements a, b of M such that $a \neq b$ holds $0 < \rho(a,b)$.

(18)[4] For every Reflexive Discerning non empty metric structure M and for all elements a, b of M holds $0 \leq \rho(a,b)$.

A non-symmetric metric space is a Reflexive Discerning triangle non empty metric structure.

Next we state two propositions:

(21)[5] For every Discerning non empty metric structure M and for all elements a, b of M such that $a \neq b$ holds $0 < \rho(a,b)$.

(23)[6] For every Reflexive Discerning non empty metric structure M and for all elements a, b of M holds $0 \leq \rho(a,b)$.

Let M be a non empty metric structure. We say that M is ultra if and only if:

(Def. 4)[7] For all elements a, b, c of M holds $\rho(a,c) \leq \max(\rho(a,b),\rho(b,c))$.

Let us observe that there exists a non empty metric structure which is strict, ultra, Reflexive, symmetric, and Discerning.

An ultra metric space is an ultra Reflexive symmetric Discerning non empty metric structure.

The following proposition is true

(26)[8] For every Discerning non empty metric structure M and for all elements a, b of M such that $a \neq b$ holds $0 < \rho(a,b)$.

One can check that every non empty metric space is Discerning.

The following proposition is true

(29)[9] For every Reflexive Discerning non empty metric structure M and for all elements a, b of M holds $0 \leq \rho(a,b)$.

Let us observe that every ultra metric space is triangle and discernible.

The function $2^2 \to 0$ from $[:\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}:]$ into $\mathbb{R}$ is defined by:

(Def. 5) $2^2 \to 0 = [:\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}:] \longmapsto 0$.

The following propositions are true:

(39)[10] $\langle\emptyset,\emptyset\rangle \in [:\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}:]$ and $\langle\emptyset,\{\emptyset\}\rangle \in [:\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}:]$ and $\langle\{\emptyset\},\emptyset\rangle \in [:\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}:]$ and $\langle\{\emptyset\},\{\emptyset\}\rangle \in [:\{\emptyset,\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}:]$.

(40) For all elements x, y of $\{\emptyset,\{\emptyset\}\}$ holds $(2^2 \to 0)(x,y) = 0$.

(42)[11] For all elements x, y of $\{\emptyset,\{\emptyset\}\}$ holds $(2^2 \to 0)(x,y) = (2^2 \to 0)(y,x)$.

[3] The proposition (15) has been removed.
[4] The proposition (17) has been removed.
[5] The propositions (19) and (20) have been removed.
[6] The proposition (22) has been removed.
[7] The definition (Def. 3) has been removed.
[8] The propositions (24) and (25) have been removed.
[9] The propositions (27) and (28) have been removed.
[10] The propositions (30)–(38) have been removed.
[11] The proposition (41) has been removed.

(43) For all elements x, y, z of $\{\emptyset, \{\emptyset\}\}$ holds $(2^2 \rightarrow 0)(x, z) \leq (2^2 \rightarrow 0)(x, y) + (2^2 \rightarrow 0)(y, z)$.

The metric structure $\ominus$ is defined by:

(Def. 6) $\ominus = \langle \{\emptyset, \{\emptyset\}\}, 2^2 \rightarrow 0 \rangle$.

One can verify that $\ominus$ is strict and non empty.

Let us mention that $\ominus$ is Reflexive, symmetric, and triangle.

Let S be a metric structure and let p, q, r be elements of S. We say that q is between p and r if and only if:

(Def. 7) $p \neq q$ and $p \neq r$ and $q \neq r$ and $\rho(p,r) = \rho(p,q) + \rho(q,r)$.

Next we state three propositions:

(47)[12] Let S be a symmetric triangle Reflexive non empty metric structure and p, q, r be elements of S. If q is between p and r, then q is between r and p.

(48) Let S be a metric space and p, q, r be elements of S. If q is between p and r, then p is not between q and r and r is not between p and q.

(49) Let S be a metric space and p, q, r, s be elements of S. Suppose q is between p and r and r is between p and s. Then q is between p and s and r is between q and s.

Let M be a non empty metric structure and let p, r be elements of M. The functor $\mathrm{IntSeg}(p,r)$ yields a subset of M and is defined by:

(Def. 8) $\mathrm{IntSeg}(p,r) = \{q; q$ ranges over elements of M: q is between p and $r\}$.

Next we state the proposition

(51)[13] For every non empty metric space M and for all elements p, r, x of M holds $x \in \mathrm{IntSeg}(p,r)$ iff x is between p and r.

Let M be a non empty metric structure and let p, r be elements of M. The functor $\mathrm{ClSeg}(p,r)$ yields a subset of M and is defined by:

(Def. 9) $\mathrm{ClSeg}(p,r) = \{q; q$ ranges over elements of M: q is between p and $r\} \cup \{p,r\}$.

The following propositions are true:

(53)[14] Let M be a non empty metric structure and p, r, x be elements of M. Then $x \in \mathrm{ClSeg}(p,r)$ if and only if one of the following conditions is satisfied:

(i) x is between p and r, or

(ii) $x = p$, or

(iii) $x = r$.

(54) For every non empty metric structure M and for all elements p, r of M holds $\mathrm{IntSeg}(p,r) \subseteq \mathrm{ClSeg}(p,r)$.

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[12] The propositions (44)–(46) have been removed.
[13] The proposition (50) has been removed.
[14] The proposition (52) has been removed.

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