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# On the Order on a Special Polygon

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**Summary.** The goal of the article is to determine the order of the special points defined in [7] on a special polygon. We restrict ourselves to the clockwise oriented finite sequences (the concept defined in this article) that start in N-min C (C being a compact non empty subset of the plane).

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The articles [16], [20], [2], [18], [5], [6], [3], [19], [4], [17], [1], [14], [15], [8], [9], [10], [11], [13], [12], and [7] provide the notation and terminology for this paper.

## 1. Preliminaries

The following propositions are true:

(1) For all sets $A$, $B$, $C$, $p$ such that $A \cap B \subseteq \{p\}$ and $p \in C$ and $C$ misses $B$ holds $A \cup C$ misses $B$.

(2) For all sets $A$, $B$, $C$, $p$ such that $A \cap C = \{p\}$ and $p \in B$ and $B \subseteq C$ holds $A \cap B = \{p\}$.

(4)[1] For all sets $A$, $B$ such that for all sets $x$, $y$ such that $x \in A$ and $y \in B$ holds $x$ misses $y$ holds $\bigcup A$ misses $\bigcup B$.

## 2. On the finite sequences

We follow the rules: $i$, $j$, $k$, $m$, $n$ denote natural numbers, $D$ denotes a non empty set, and $f$ denotes a finite sequence of elements of $D$.

We now state several propositions:

(5) If $i \leq j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f,i,j)$, then $(k+i) -' 1 \in \operatorname{dom} f$.

(6) If $i > j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f,i,j)$, then $(i -' k) + 1 \in \operatorname{dom} f$.

(7) If $i \leq j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f,i,j)$, then $(\operatorname{mid}(f,i,j))_k = f_{(k+i)-'1}$.

(8) If $i > j$ and $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $k \in \operatorname{dom} \operatorname{mid}(f,i,j)$, then $(\operatorname{mid}(f,i,j))_k = f_{(i-'k)+1}$.

(9) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{len} \operatorname{mid}(f,i,j) \geq 1$.

[1] The proposition (3) has been removed.

(10) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{len} \operatorname{mid}(f,i,j) = 1$, then $i = j$.

(11) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{mid}(f,i,j)$ is non empty.

(12) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $(\operatorname{mid}(f,i,j))_1 = f_i$.

(13) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $(\operatorname{mid}(f,i,j))_{\operatorname{len} \operatorname{mid}(f,i,j)} = f_j$.

## 3. Compact subsets of the plane

In the sequel $X$ denotes a compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following four propositions:

(14) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{2}} = \text{N-bound}(X)$ holds $p \in \mathrm{N}_{\mathrm{most}}(X)$.

(15) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{2}} = \text{S-bound}(X)$ holds $p \in \mathrm{S}_{\mathrm{most}}(X)$.

(16) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{1}} = \text{W-bound}(X)$ holds $p \in \mathrm{W}_{\mathrm{most}}(X)$.

(17) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ and $p_{\mathbf{1}} = \text{E-bound}(X)$ holds $p \in \mathrm{E}_{\mathrm{most}}(X)$.

## 4. Finite sequences on the plane

One can prove the following propositions:

(18) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leq i$ and $i \leq j$ and $j \leq \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(\operatorname{mid}(f,i,j)) = \bigcup\{\mathcal{L}(f,k) : i \leq k \wedge k < j\}$.

(19) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{dom} \mathbf{X}\text{-coordinate}(f) = \operatorname{dom} f$.

(20) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{dom} \mathbf{Y}\text{-coordinate}(f) = \operatorname{dom} f$.

(21) For all points $a$, $b$, $c$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{1}} \leq b_{\mathbf{1}}$ and $c_{\mathbf{1}} \leq b_{\mathbf{1}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{1}} = b_{\mathbf{1}}$ and $c_{\mathbf{1}} = b_{\mathbf{1}}$.

(22) For all points $a$, $b$, $c$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{2}} \leq b_{\mathbf{2}}$ and $c_{\mathbf{2}} \leq b_{\mathbf{2}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{2}} = b_{\mathbf{2}}$ and $c_{\mathbf{2}} = b_{\mathbf{2}}$.

(23) For all points $a$, $b$, $c$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{1}} \geq b_{\mathbf{1}}$ and $c_{\mathbf{1}} \geq b_{\mathbf{1}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{1}} = b_{\mathbf{1}}$ and $c_{\mathbf{1}} = b_{\mathbf{1}}$.

(24) For all points $a$, $b$, $c$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $b \in \mathcal{L}(a,c)$ and $a_{\mathbf{2}} \geq b_{\mathbf{2}}$ and $c_{\mathbf{2}} \geq b_{\mathbf{2}}$ holds $a = b$ or $b = c$ or $a_{\mathbf{2}} = b_{\mathbf{2}}$ and $c_{\mathbf{2}} = b_{\mathbf{2}}$.

## 5. The area of a sequence

Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. We say that $g$ is in the area of $f$ if and only if:

(Def. 1) For every $n$ such that $n \in \operatorname{dom} g$ holds $\text{W-bound}(\widetilde{\mathcal{L}}(f)) \leq (g_n)_{\mathbf{1}}$ and $(g_n)_{\mathbf{1}} \leq \text{E-bound}(\widetilde{\mathcal{L}}(f))$ and $\text{S-bound}(\widetilde{\mathcal{L}}(f)) \leq (g_n)_{\mathbf{2}}$ and $(g_n)_{\mathbf{2}} \leq \text{N-bound}(\widetilde{\mathcal{L}}(f))$.

The following propositions are true:

(25) Every non trivial finite sequence $f$ of elements of $\mathcal{E}^2_{\mathrm{T}}$ is in the area of $f$.

(26) Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $g$ is in the area of $f$. Let given $i$, $j$. If $i \in \operatorname{dom} g$ and $j \in \operatorname{dom} g$, then $\operatorname{mid}(g,i,j)$ is in the area of $f$.

(27) Let $f$ be a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given $i$, $j$. If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$, then $\operatorname{mid}(f,i,j)$ is in the area of $f$.

(28) Let $f$, $g$, $h$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $g$ is in the area of $f$ and $h$ is in the area of $f$. Then $g \frown h$ is in the area of $f$.

(29) For every non trivial finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{NE-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is in the area of $f$.

(30) For every non trivial finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{NW-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is in the area of $f$.

(31) For every non trivial finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{SE-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is in the area of $f$.

(32) For every non trivial finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ holds $\langle \text{SW-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is in the area of $f$.

## 6. Horizontal and vertical connections

Let $f$, $g$ be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. We say that $g$ is a h.c. for $f$ if and only if:

(Def. 2) $g$ is in the area of $f$ and $(g_1)_{\mathbf{1}} = \text{W-bound}(\widetilde{\mathcal{L}}(f))$ and $(g_{\operatorname{len} g})_{\mathbf{1}} = \text{E-bound}(\widetilde{\mathcal{L}}(f))$.

We say that $g$ is a v.c. for $f$ if and only if:

(Def. 3) $g$ is in the area of $f$ and $(g_1)_{\mathbf{2}} = \text{S-bound}(\widetilde{\mathcal{L}}(f))$ and $(g_{\operatorname{len} g})_{\mathbf{2}} = \text{N-bound}(\widetilde{\mathcal{L}}(f))$.

Next we state the proposition

(33) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and $g$, $h$ be one-to-one special finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose $2 \leq \operatorname{len} g$ and $2 \leq \operatorname{len} h$ and $g$ is a h.c. for $f$ and $h$ is a v.c. for $f$. Then $\widetilde{\mathcal{L}}(g)$ meets $\widetilde{\mathcal{L}}(h)$.

## 7. Orientation

Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. We say that $f$ is clockwise oriented if and only if:

(Def. 4) $(f \circlearrowleft \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f)))_2 \in \mathrm{N}_{\text{most}}(\widetilde{\mathcal{L}}(f))$.

Next we state the proposition

(34) Let $f$ be a non constant standard special circular sequence. If $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$, then $f$ is clockwise oriented iff $f_2 \in \mathrm{N}_{\text{most}}(\widetilde{\mathcal{L}}(f))$.

Let us observe that $\square_{\mathcal{E}^2}$ is compact.

Next we state several propositions:

(35) $\text{N-bound}(\square_{\mathcal{E}^2}) = 1$.

(36) $\text{W-bound}(\square_{\mathcal{E}^2}) = 0$.

(37) $\text{E-bound}(\square_{\mathcal{E}^2}) = 1$.

(38) $\text{S-bound}(\square_{\mathcal{E}^2}) = 0$.

(39) $\mathrm{N}_{\text{most}}(\square_{\mathcal{E}^2}) = \mathcal{L}([0,1],[1,1])$.

(40) $\mathrm{N}_{\min}(\square_{\mathcal{E}^2}) = [0,1]$.

Let $X$ be a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\rm T}$. Note that $\operatorname{SpStSeq} X$ is clockwise oriented.

One can verify that there exists a non constant standard special circular sequence which is clockwise oriented.

We now state two propositions:

(41) Let $f$ be a non constant standard special circular sequence and given $i$, $j$. Suppose $i > j$ but $1 < j$ and $i \leq \operatorname{len} f$ or $1 \leq j$ and $i < \operatorname{len} f$. Then $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$.

(42) Let $f$ be a non constant standard special circular sequence and given $i$, $j$. Suppose $i < j$ but $1 < i$ and $j \leq \operatorname{len} f$ or $1 \leq i$ and $j < \operatorname{len} f$. Then $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$.

In the sequel $f$ denotes a non trivial finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.
The following propositions are true:

(43) $\mathrm{N}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(44) $\mathrm{N}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(45) $\mathrm{S}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(46) $\mathrm{S}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(47) $\mathrm{W}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(48) $\mathrm{W}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(49) $\mathrm{E}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

(50) $\mathrm{E}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)) \in \operatorname{rng} f$.

In the sequel $f$ denotes a non constant standard special circular sequence.
We now state a number of propositions:

(51) If $1 \leq i$ and $i \leq j$ and $j < m$ and $m \leq n$ and $n \leq \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f,i,j))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f,m,n))$.

(52) If $1 \leq i$ and $i \leq j$ and $j < m$ and $m \leq n$ and $n \leq \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f,i,j))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f,n,m))$.

(53) If $1 \leq i$ and $i \leq j$ and $j < m$ and $m \leq n$ and $n \leq \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f,j,i))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f,n,m))$.

(54) If $1 \leq i$ and $i \leq j$ and $j < m$ and $m \leq n$ and $n \leq \operatorname{len} f$ and $1 < i$ or $n < \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f,j,i))$ misses $\widetilde{\mathcal{L}}(\operatorname{mid}(f,m,n))$.

(55) $(\mathrm{N}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}} < (\mathrm{N}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}}$.

(56) $\mathrm{N}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{N}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f))$.

(57) $(\mathrm{E}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}} < (\mathrm{E}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}}$.

(58) $\mathrm{E}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{E}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f))$.

(59) $(\mathrm{S}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}} < (\mathrm{S}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{1}}$.

(60) $\mathrm{S}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{S}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f))$.

(61) $(\mathrm{W}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}} < (\mathrm{W}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)))_{\mathbf{2}}$.

(62) $\mathrm{W}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)) \neq \mathrm{W}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f))$.

(63) $\mathcal{L}(\text{NW-corner}(\widetilde{\mathcal{L}}(f)), \mathrm{N}_{\mathrm{min}}(\widetilde{\mathcal{L}}(f)))$ misses $\mathcal{L}(\mathrm{N}_{\mathrm{max}}(\widetilde{\mathcal{L}}(f)), \text{NE-corner}(\widetilde{\mathcal{L}}(f)))$.

(64) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \neq f_1$ and $p_{\mathbf{1}} = (f_1)_{\mathbf{1}}$ or $p_{\mathbf{2}} = (f_1)_{\mathbf{2}}$ and $\mathcal{L}(p, f_1) \cap \widetilde{\mathcal{L}}(f) = \{f_1\}$. Then $\langle p \rangle \frown f$ is a S-sequence in $\mathbb{R}^2$.

(65) Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f$ is a special sequence and $p \neq f_{\operatorname{len} f}$ and $p_{\mathbf{1}} = (f_{\operatorname{len} f})_{\mathbf{1}}$ or $p_{\mathbf{2}} = (f_{\operatorname{len} f})_{\mathbf{2}}$ and $\mathcal{L}(p, f_{\operatorname{len} f}) \cap \widetilde{\mathcal{L}}(f) = \{f_{\operatorname{len} f}\}$. Then $f \frown \langle p \rangle$ is a S-sequence in $\mathbb{R}^2$.

## 8. Appending corners

The following propositions are true:

(66) Let given $i$, $j$. Suppose $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$ and $f_j = \mathrm{N}_{\max}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{N}_{\max}(\widetilde{\mathcal{L}}(f)) \neq \text{NE-corner}(\widetilde{\mathcal{L}}(f))$. Then $(\operatorname{mid}(f,i,j)) \frown \langle \text{NE-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(67) Let given $i$, $j$. Suppose $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$ and $f_j = \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{E}_{\max}(\widetilde{\mathcal{L}}(f)) \neq \text{NE-corner}(\widetilde{\mathcal{L}}(f))$. Then $(\operatorname{mid}(f,i,j)) \frown \langle \text{NE-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(68) Let given $i$, $j$. Suppose $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$ and $f_j = \mathrm{S}_{\max}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{S}_{\max}(\widetilde{\mathcal{L}}(f)) \neq \text{SE-corner}(\widetilde{\mathcal{L}}(f))$. Then $(\operatorname{mid}(f,i,j)) \frown \langle \text{SE-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(69) Let given $i$, $j$. Suppose $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$ and $f_j = \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{E}_{\max}(\widetilde{\mathcal{L}}(f)) \neq \text{NE-corner}(\widetilde{\mathcal{L}}(f))$. Then $(\operatorname{mid}(f,i,j)) \frown \langle \text{NE-corner}(\widetilde{\mathcal{L}}(f)) \rangle$ is a S-sequence in $\mathbb{R}^2$.

(70) Let given $i$, $j$. Suppose $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$ and $f_i = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f)) \neq \text{NW-corner}(\widetilde{\mathcal{L}}(f))$. Then $\langle \text{NW-corner}(\widetilde{\mathcal{L}}(f)) \rangle \frown \operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$.

(71) Let given $i$, $j$. Suppose $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $\operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$ and $f_i = \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(f))$ and $\mathrm{W}_{\min}(\widetilde{\mathcal{L}}(f)) \neq \text{SW-corner}(\widetilde{\mathcal{L}}(f))$. Then $\langle \text{SW-corner}(\widetilde{\mathcal{L}}(f)) \rangle \frown \operatorname{mid}(f,i,j)$ is a S-sequence in $\mathbb{R}^2$.

Let $f$ be a non constant standard special circular sequence. Note that $\widetilde{\mathcal{L}}(f)$ satisfies conditions of simple closed curve.

## 9. The order

Next we state two propositions:

(72) If $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; f < (\mathrm{N}_{\max}(\widetilde{\mathcal{L}}(f))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; f$.

(73) If $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{N}_{\max}(\widetilde{\mathcal{L}}(f))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; f > 1$.

In the sequel $z$ is a clockwise oriented non constant standard special circular sequence.

The following propositions are true:

(74) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N}_{\max}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{E}_{\max}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{N}_{\max}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < (\mathrm{E}_{\max}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z$.

(75) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E}_{\max}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < (\mathrm{E}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z$.

(76) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{E}_{\min}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{S}_{\max}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{E}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < (\mathrm{S}_{\max}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z$.

(77) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S}_{\max}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < (\mathrm{S}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z$.

(78) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{S}_{\min}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W}_{\min}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{S}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < (\mathrm{W}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z$.

(79) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$ and $\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z)) \neq \mathrm{W}_{\max}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < (\mathrm{W}_{\max}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z$.

(80) If $z_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(z))$, then $(\mathrm{W}_{\min}(\widetilde{\mathcal{L}}(z))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; z < \operatorname{len} z$.

(81) If $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$, then $(\mathrm{W}_{\max}(\widetilde{\mathcal{L}}(f))) \leftarrow\!\!\!\!\!\!\!\!\;\;\; f < \operatorname{len} f$.

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