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Semilattice Operations on Finite Subsets

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Summary. In the article we deal with a binary operation that is associative, commutative. We define for such an operation a functor that depends on two more arguments: a finite set of indices and a function indexing elements of the domain of the operation and yields the result of applying the operation to all indexed elements. The definition has a restriction that requires that either the set of indices is non empty or the operation has the unity. We prove theorems describing some properties of the functor introduced. Most of them we prove in two versions depending on which requirement is fulfilled. In the second part we deal with the union of finite sets that enjoys mentioned above properties. We prove analogs of the theorems proved in the first part. We precede the main part of the article with auxiliary theorems related to boolean properties of sets, enumerated sets, finite subsets, and functions. We define a casting function that yields to a set the empty set typed as a finite subset of the set. We prove also two schemes of the induction on finite sets.

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The articles [6], [5], [8], [9], [2], [7], [3], [1], and [4] provide the notation and terminology for this paper.

In this paper x, y, X, Y denote sets.

The following propositions are true:

(3)[1] $\{x\} \subseteq \{x,y,z\}$.

(4) $\{x,y\} \subseteq \{x,y,z\}$.

(5) If $X \subseteq Y \cup \{x\}$, then $x \in X$ or $X \subseteq Y$.

(6) $x \in X \cup \{y\}$ iff $x \in X$ or $x = y$.

(8)[2] $X \cup \{x\} \subseteq Y$ iff $x \in Y$ and $X \subseteq Y$.

(11)[3] For all X, Y and for every function f holds $f^\circ(Y \setminus f^{-1}(X)) = f^\circ Y \setminus X$.

In the sequel X, Y are non empty sets and f is a function from X into Y.

The following two propositions are true:

(12) For every element x of X holds $x \in f^{-1}(\{f(x)\})$.

(13) For every element x of X holds $f^\circ\{x\} = \{f(x)\}$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (7) has been removed.
[3] The propositions (9) and (10) have been removed.

The scheme *SubsetEx* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a subset B of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in B$ iff $\mathcal{P}[x]$

for all values of the parameters.

Next we state several propositions:

(14) For every element B of $\operatorname{Fin} X$ and for every x such that $x \in B$ holds x is an element of X.

(15) Let A be an element of $\operatorname{Fin} X$, B be a set, and f be a function from X into Y. If for every element x of X such that $x \in A$ holds $f(x) \in B$, then $f^{\circ}A \subseteq B$.

(16) For every set X and for every element B of $\operatorname{Fin} X$ and for every set A such that $A \subseteq B$ holds A is an element of $\operatorname{Fin} X$.

(18)[4] For every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ there exists an element x of X such that $x \in B$.

(19) For every element A of $\operatorname{Fin} X$ such that $f^{\circ}A = \emptyset$ holds $A = \emptyset$.

Let X be a set. Note that there exists an element of $\operatorname{Fin} X$ which is empty.

Let X be a set. The functor $\emptyset_X$ yields an empty element of $\operatorname{Fin} X$ and is defined by:

(Def. 1) $\emptyset_X = \emptyset$.

The scheme *FinSubFuncEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\operatorname{Fin} \mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{A}$ into $\operatorname{Fin} \mathcal{A}$ such that for all elements b, a of $\mathcal{A}$ holds $a \in f(b)$ iff $a \in \mathcal{B}$ and $\mathcal{P}[a,b]$

for all values of the parameters.

Let X be a non empty set and let F be a binary operation on X. We say that F is unital if and only if:

(Def. 2) There exists an element of X which is a unity w.r.t. F.

We introduce F has a unity as a synonym of F is unital.

Next we state two propositions:

(22)[5] For every non empty set X and for every binary operation F on X holds F has a unity iff $\mathbf{1}_F$ is a unity w.r.t. F.

(23) Let X be a non empty set and F be a binary operation on X. If F has a unity, then for every element x of X holds $F(\mathbf{1}_F, x) = x$ and $F(x, \mathbf{1}_F) = x$.

Let X be a non empty set. One can check that there exists an element of $\operatorname{Fin} X$ which is non empty.

Let X be a non empty set and let x be an element of X. Then $\{x\}$ is an element of $\operatorname{Fin} X$. Let y be an element of X. Then $\{x,y\}$ is an element of $\operatorname{Fin} X$. Let z be an element of X. Then $\{x,y,z\}$ is an element of $\operatorname{Fin} X$.

Let X be a set and let A, B be elements of $\operatorname{Fin} X$. Then $A \cup B$ is an element of $\operatorname{Fin} X$.

Let X be a set and let A, B be elements of $\operatorname{Fin} X$. Then $A \setminus B$ is an element of $\operatorname{Fin} X$.

Now we present three schemes. The scheme *FinSubInd1* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element B of $\operatorname{Fin} \mathcal{A}$ holds $\mathcal{P}[B]$

provided the following conditions are satisfied:

- $\mathcal{P}[\emptyset_{\mathcal{A}}]$, and
- For every element B' of $\operatorname{Fin} \mathcal{A}$ and for every element b of $\mathcal{A}$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$.

The scheme *FinSubInd2* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element B of $\operatorname{Fin} \mathcal{A}$ such that $B \neq \emptyset$ holds $\mathcal{P}[B]$

[4] The proposition (17) has been removed.

[5] The propositions (20) and (21) have been removed.

provided the parameters meet the following requirements:

- For every element x of $\mathcal{A}$ holds $\mathcal{P}[\{x\}]$, and
- For all elements B_1, B_2 of Fin $\mathcal{A}$ such that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$ holds if $\mathcal{P}[B_1]$ and $\mathcal{P}[B_2]$, then $\mathcal{P}[B_1 \cup B_2]$.

The scheme *FinSubInd3* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element B of Fin $\mathcal{A}$ holds $\mathcal{P}[B]$

provided the following conditions are satisfied:

- $\mathcal{P}[\emptyset_{\mathcal{A}}]$, and
- For every element B' of Fin $\mathcal{A}$ and for every element b of $\mathcal{A}$ such that $\mathcal{P}[B']$ holds $\mathcal{P}[B' \cup \{b\}]$.

Let X, Y be non empty sets, let F be a binary operation on Y, let B be an element of Fin X, and let f be a function from X into Y. Let us assume that $B \neq \emptyset$ or F has a unity F is commutative and F is associative. The functor $F\text{-}\sum_B f$ yielding an element of Y is defined by the condition (Def. 3).

(Def. 3) There exists a function G from Fin X into Y such that

(i) $F\text{-}\sum_B f = G(B)$,

(ii) for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$,

(iii) for every element x of X holds $G(\{x\}) = f(x)$, and

(iv) for every element B' of Fin X such that $B' \subseteq B$ and $B' \neq \emptyset$ and for every element x of X such that $x \in B \setminus B'$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

Next we state the proposition

(25)[6] Let X, Y be non empty sets, F be a binary operation on Y, B be an element of Fin X, and f be a function from X into Y. Suppose $B \neq \emptyset$ or F has a unity but F is idempotent, commutative, and associative. Let I_1 be an element of Y. Then $I_1 = F\text{-}\sum_B f$ if and only if there exists a function G from Fin X into Y such that $I_1 = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of Fin X such that $B' \subseteq B$ and $B' \neq \emptyset$ and for every element x of X such that $x \in B$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

For simplicity, we adopt the following rules: X, Y are non empty sets, F is a binary operation on Y, B is an element of Fin X, and f is a function from X into Y.

We now state a number of propositions:

(26) If F is commutative and associative, then for every element b of X holds $F\text{-}\sum_{\{b\}} f = f(b)$.

(27) If F is idempotent, commutative, and associative, then for all elements a, b of X holds $F\text{-}\sum_{\{a,b\}} f = F(f(a), f(b))$.

(28) Suppose F is idempotent, commutative, and associative. Let a, b, c be elements of X. Then $F\text{-}\sum_{\{a,b,c\}} f = F(F(f(a), f(b)), f(c))$.

(29) Suppose F is idempotent, commutative, and associative and $B \neq \emptyset$. Let x be an element of X. Then $F\text{-}\sum_{B \cup \{x\}} f = F(F\text{-}\sum_B f, f(x))$.

(30) Suppose F is idempotent, commutative, and associative. Let B_1, B_2 be elements of Fin X. If $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$, then $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f, F\text{-}\sum_{B_2} f)$.

(31) Suppose F is commutative, associative, and idempotent. Let x be an element of X. If $x \in B$, then $F(f(x), F\text{-}\sum_B f) = F\text{-}\sum_B f$.

(32) Suppose F is commutative, associative, and idempotent. Let B, C be elements of Fin X. If $B \neq \emptyset$ and $B \subseteq C$, then $F(F\text{-}\sum_B f, F\text{-}\sum_C f) = F\text{-}\sum_C f$.

(33) Suppose $B \neq \emptyset$ and F is commutative, associative, and idempotent. Let a be an element of Y. If for every element b of X such that $b \in B$ holds $f(b) = a$, then $F\text{-}\sum_B f = a$.

[6] The proposition (24) has been removed.

(34) Suppose F is commutative, associative, and idempotent. Let a be an element of Y. If $f^\circ B = \{a\}$, then $F\text{-}\sum_B f = a$.

(35) Suppose F is commutative, associative, and idempotent. Let f, g be functions from X into Y and A, B be elements of $\mathrm{Fin}\,X$. If $A \neq \emptyset$ and $f^\circ A = g^\circ B$, then $F\text{-}\sum_A f = F\text{-}\sum_B g$.

(36) Let F, G be binary operations on Y. Suppose F is idempotent, commutative, and associative and G is distributive w.r.t. F. Let B be an element of $\mathrm{Fin}\,X$. Suppose $B \neq \emptyset$. Let f be a function from X into Y and a be an element of Y. Then $G(a, F\text{-}\sum_B f) = F\text{-}\sum_B G^\circ(a, f)$.

(37) Let F, G be binary operations on Y. Suppose F is idempotent, commutative, and associative and G is distributive w.r.t. F. Let B be an element of $\mathrm{Fin}\,X$. Suppose $B \neq \emptyset$. Let f be a function from X into Y and a be an element of Y. Then $G(F\text{-}\sum_B f, a) = F\text{-}\sum_B G^\circ(f, a)$.

Let X, Y be non empty sets, let f be a function from X into Y, and let A be an element of $\mathrm{Fin}\,X$. Then $f^\circ A$ is an element of $\mathrm{Fin}\,Y$.

Next we state several propositions:

(38) Let A, X, Y be non empty sets and F be a binary operation on A. Suppose F is idempotent, commutative, and associative. Let B be an element of $\mathrm{Fin}\,X$. Suppose $B \neq \emptyset$. Let f be a function from X into Y and g be a function from Y into A. Then $F\text{-}\sum_{f^\circ B} g = F\text{-}\sum_B g \cdot f$.

(39) Suppose F is commutative, associative, and idempotent. Let Z be a non empty set and G be a binary operation on Z. Suppose G is commutative, associative, and idempotent. Let f be a function from X into Y and g be a function from Y into Z. Suppose that for all elements x, y of Y holds $g(F(x, y)) = G(g(x), g(y))$. Let B be an element of $\mathrm{Fin}\,X$. If $B \neq \emptyset$, then $g(F\text{-}\sum_B f) = G\text{-}\sum_B g \cdot f$.

(40) If F is commutative and associative and has a unity, then for every f holds $F\text{-}\sum_{\emptyset_X} f = \mathbf{1}_F$.

(41) Suppose F is idempotent, commutative, and associative and has a unity. Let x be an element of X. Then $F\text{-}\sum_{B \cup \{x\}} f = F(F\text{-}\sum_B f, f(x))$.

(42) Suppose F is idempotent, commutative, and associative and has a unity. Let B_1, B_2 be elements of $\mathrm{Fin}\,X$. Then $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f, F\text{-}\sum_{B_2} f)$.

(43) Suppose F is commutative, associative, and idempotent and has a unity. Let f, g be functions from X into Y and A, B be elements of $\mathrm{Fin}\,X$. If $f^\circ A = g^\circ B$, then $F\text{-}\sum_A f = F\text{-}\sum_B g$.

(44) Let A, X, Y be non empty sets and F be a binary operation on A. Suppose F is idempotent, commutative, and associative and has a unity. Let B be an element of $\mathrm{Fin}\,X$, f be a function from X into Y, and g be a function from Y into A. Then $F\text{-}\sum_{f^\circ B} g = F\text{-}\sum_B g \cdot f$.

(45) Suppose F is commutative, associative, and idempotent and has a unity. Let Z be a non empty set and G be a binary operation on Z. Suppose G is commutative, associative, and idempotent and has a unity. Let f be a function from X into Y and g be a function from Y into Z. Suppose $g(\mathbf{1}_F) = \mathbf{1}_G$ and for all elements x, y of Y holds $g(F(x, y)) = G(g(x), g(y))$. Let B be an element of $\mathrm{Fin}\,X$. Then $g(F\text{-}\sum_B f) = G\text{-}\sum_B g \cdot f$.

Let A be a set. The functor $\mathrm{FinUnion}_A$ yields a binary operation on $\mathrm{Fin}\,A$ and is defined as follows:

(Def. 4) For all elements x, y of $\mathrm{Fin}\,A$ holds $\mathrm{FinUnion}_A(x, y) = x \cup y$.

In the sequel A is a set.

The following propositions are true:

(49)[7] $\mathrm{FinUnion}_A$ is idempotent.

(50) $\mathrm{FinUnion}_A$ is commutative.

[7] The propositions (46)–(48) have been removed.

(51) $\mathrm{FinUnion}_A$ is associative.

(52) $\emptyset_A$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(53) $\mathrm{FinUnion}_A$ has a unity.

(54) $\mathbf{1}_{\mathrm{FinUnion}_A}$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(55) $\mathbf{1}_{\mathrm{FinUnion}_A} = \emptyset$.

For simplicity, we adopt the following convention: X, Y denote non empty sets, A denotes a set, f denotes a function from X into $\mathrm{Fin}A$, and i, j, k denote elements of X.

Let X be a non empty set, let A be a set, let B be an element of $\mathrm{Fin}X$, and let f be a function from X into $\mathrm{Fin}A$. The functor $\mathrm{FinUnion}(B, f)$ yielding an element of $\mathrm{Fin}A$ is defined as follows:

(Def. 5) $\mathrm{FinUnion}(B, f) = \mathrm{FinUnion}_A\text{-}\sum_B f$.

We now state a number of propositions:

(56) $\mathrm{FinUnion}(\{i\}, f) = f(i)$.

(57) $\mathrm{FinUnion}(\{i, j\}, f) = f(i) \cup f(j)$.

(58) $\mathrm{FinUnion}(\{i, j, k\}, f) = f(i) \cup f(j) \cup f(k)$.

(59) $\mathrm{FinUnion}(\emptyset_X, f) = \emptyset$.

(60) For every element B of $\mathrm{Fin}X$ holds $\mathrm{FinUnion}(B \cup \{i\}, f) = \mathrm{FinUnion}(B, f) \cup f(i)$.

(61) For every element B of $\mathrm{Fin}X$ holds $\mathrm{FinUnion}(B, f) = \bigcup(f^\circ B)$.

(62) For all elements B_1, B_2 of $\mathrm{Fin}X$ holds $\mathrm{FinUnion}(B_1 \cup B_2, f) = \mathrm{FinUnion}(B_1, f) \cup \mathrm{FinUnion}(B_2, f)$.

(63) Let B be an element of $\mathrm{Fin}X$, f be a function from X into Y, and g be a function from Y into $\mathrm{Fin}A$. Then $\mathrm{FinUnion}(f^\circ B, g) = \mathrm{FinUnion}(B, g \cdot f)$.

(64) Let A, X be non empty sets, Y be a set, and G be a binary operation on A. Suppose G is commutative, associative, and idempotent. Let B be an element of $\mathrm{Fin}X$. Suppose $B \neq \emptyset$. Let f be a function from X into $\mathrm{Fin}Y$ and g be a function from $\mathrm{Fin}Y$ into A. Suppose that for all elements x, y of $\mathrm{Fin}Y$ holds $g(x \cup y) = G(g(x), g(y))$. Then $g(\mathrm{FinUnion}(B, f)) = G\text{-}\sum_B g \cdot f$.

(65) Let Z be a non empty set, Y be a set, and G be a binary operation on Z. Suppose G is commutative, associative, and idempotent and has a unity. Let f be a function from X into $\mathrm{Fin}Y$ and g be a function from $\mathrm{Fin}Y$ into Z. Suppose $g(\emptyset_Y) = \mathbf{1}_G$ and for all elements x, y of $\mathrm{Fin}Y$ holds $g(x \cup y) = G(g(x), g(y))$. Let B be an element of $\mathrm{Fin}X$. Then $g(\mathrm{FinUnion}(B, f)) = G\text{-}\sum_B g \cdot f$.

Let A be a set. The functor $\mathrm{singleton}_A$ yields a function from A into $\mathrm{Fin}A$ and is defined by:

(Def. 6) For every set x such that $x \in A$ holds $\mathrm{singleton}_A(x) = \{x\}$.

Next we state several propositions:

(67)[8] Let A be a non empty set and f be a function from A into $\mathrm{Fin}A$. Then $f = \mathrm{singleton}_A$ if and only if for every element x of A holds $f(x) = \{x\}$.

(68) For every set x and for every element y of X holds $x \in \mathrm{singleton}_X(y)$ iff $x = y$.

(69) For all elements x, y, z of X such that $x \in \mathrm{singleton}_X(z)$ and $y \in \mathrm{singleton}_X(z)$ holds $x = y$.

(70) Let B be an element of $\mathrm{Fin}X$ and x be a set. Then $x \in \mathrm{FinUnion}(B, f)$ if and only if there exists an element i of X such that $i \in B$ and $x \in f(i)$.

[8] The proposition (66) has been removed.

(71) For every element B of $\mathrm{Fin}\,X$ holds $\mathrm{FinUnion}(B, \mathrm{singleton}_X) = B$.

(72) Let Y, Z be sets, f be a function from X into $\mathrm{Fin}\,Y$, and g be a function from $\mathrm{Fin}\,Y$ into $\mathrm{Fin}\,Z$. Suppose $g(\emptyset_Y) = \emptyset_Z$ and for all elements x, y of $\mathrm{Fin}\,Y$ holds $g(x \cup y) = g(x) \cup g(y)$. Let B be an element of $\mathrm{Fin}\,X$. Then $g(\mathrm{FinUnion}(B, f)) = \mathrm{FinUnion}(B, g \cdot f)$.

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