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Families of Sets

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Summary. The article contains definitions of the following concepts: family of sets, family of subsets of a set, the intersection of a family of sets. Functors $\cup$, $\cap$, and $\setminus$ are redefined for families of subsets of a set. Some properties of these notions are presented.

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The articles [2], [1], and [3] provide the notation and terminology for this paper.

In this paper X, Y, Z, Z_1, D, x denote sets.

Let us consider X. The functor $\bigcap X$ is defined by:

(Def. 1)(i) For every x holds $x \in \bigcap X$ iff for every Y such that $Y \in X$ holds $x \in Y$ if $X \neq \emptyset$,

(ii) $\bigcap X = \emptyset$, otherwise.

Next we state a number of propositions:

(2)[1] $\bigcap \emptyset = \emptyset$.

(3) $\bigcap X \subseteq \bigcup X$.

(4) If $Z \in X$, then $\bigcap X \subseteq Z$.

(5) If $\emptyset \in X$, then $\bigcap X = \emptyset$.

(6) If $X \neq \emptyset$ and for every Z_1 such that $Z_1 \in X$ holds $Z \subseteq Z_1$, then $Z \subseteq \bigcap X$.

(7) If $X \neq \emptyset$ and $X \subseteq Y$, then $\bigcap Y \subseteq \bigcap X$.

(8) If $X \in Y$ and $X \subseteq Z$, then $\bigcap Y \subseteq Z$.

(9) If $X \in Y$ and X misses Z, then $\bigcap Y$ misses Z.

(10) If $X \neq \emptyset$ and $Y \neq \emptyset$, then $\bigcap (X \cup Y) = \bigcap X \cap \bigcap Y$.

(11) $\bigcap \{x\} = x$.

(12) $\bigcap \{X, Y\} = X \cap Y$.

In the sequel S_1, S_2, S_3 denote sets.

Let us consider S_1, S_2. We say that S_1 is finer than S_2 if and only if:

(Def. 2) For every X such that $X \in S_1$ there exists Y such that $Y \in S_2$ and $X \subseteq Y$.

[1] The proposition (1) has been removed.

Let us note that the predicate S_1 is finer than S_2 is reflexive. We say that S_2 is coarser than S_1 if and only if:

(Def. 3) For every Y such that $Y \in S_2$ there exists X such that $X \in S_1$ and $X \subseteq Y$.

Let us note that the predicate S_2 is coarser than S_1 is reflexive.

We now state several propositions:

(17)[2] If $S_1 \subseteq S_2$, then S_1 is finer than S_2.

(18) If S_1 is finer than S_2, then $\bigcup S_1 \subseteq \bigcup S_2$.

(19) If $S_2 \neq \emptyset$ and S_2 is coarser than S_1, then $\bigcap S_1 \subseteq \bigcap S_2$.

(20) $\emptyset$ is finer than S_1.

(21) If S_1 is finer than $\emptyset$, then $S_1 = \emptyset$.

(23)[3] If S_1 is finer than S_2 and S_2 is finer than S_3, then S_1 is finer than S_3.

(24) If S_1 is finer than $\{Y\}$, then for every X such that $X \in S_1$ holds $X \subseteq Y$.

(25) If S_1 is finer than $\{X, Y\}$, then for every Z such that $Z \in S_1$ holds $Z \subseteq X$ or $Z \subseteq Y$.

Let us consider S_1, S_2. The functor $S_1 \Cup S_2$ is defined as follows:

(Def. 4) $Z \in S_1 \Cup S_2$ iff there exist X, Y such that $X \in S_1$ and $Y \in S_2$ and $Z = X \cup Y$.

Let us note that the functor $S_1 \Cup S_2$ is commutative. The functor $S_1 \Cap S_2$ is defined as follows:

(Def. 5) $Z \in S_1 \Cap S_2$ iff there exist X, Y such that $X \in S_1$ and $Y \in S_2$ and $Z = X \cap Y$.

Let us notice that the functor $S_1 \Cap S_2$ is commutative. The functor $S_1 \setminus\setminus S_2$ is defined by:

(Def. 6) $Z \in S_1 \setminus\setminus S_2$ iff there exist X, Y such that $X \in S_1$ and $Y \in S_2$ and $Z = X \setminus Y$.

We now state a number of propositions:

(29)[4] S_1 is finer than $S_1 \Cup S_1$.

(30) $S_1 \Cap S_1$ is finer than S_1.

(31) $S_1 \setminus\setminus S_1$ is finer than S_1.

(34)[5] If S_1 meets S_2, then $\bigcap S_1 \cap \bigcap S_2 = \bigcap (S_1 \Cap S_2)$.

(35) If $S_2 \neq \emptyset$, then $X \cup \bigcap S_2 = \bigcap (\{X\} \Cup S_2)$.

(36) $X \cap \bigcup S_2 = \bigcup (\{X\} \Cap S_2)$.

(37) If $S_2 \neq \emptyset$, then $X \setminus \bigcup S_2 = \bigcap (\{X\} \setminus\setminus S_2)$.

(38) If $S_2 \neq \emptyset$, then $X \setminus \bigcap S_2 = \bigcup (\{X\} \setminus\setminus S_2)$.

(39) $\bigcup (S_1 \Cap S_2) \subseteq \bigcup S_1 \cap \bigcup S_2$.

(40) If $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, then $\bigcap S_1 \cup \bigcap S_2 \subseteq \bigcap (S_1 \Cup S_2)$.

(41) $\bigcap (S_1 \setminus\setminus S_2) \subseteq \bigcap S_1 \setminus \bigcap S_2$.

Let D be a set. Family of subsets of D is defined by:

[2] The propositions (13)–(16) have been removed.
[3] The proposition (22) has been removed.
[4] The propositions (26)–(28) have been removed.
[5] The propositions (32) and (33) have been removed.

(Def. 7) It $\subseteq 2^D$.

Let D be a set. We see that the family of subsets of D is a subset of 2^D.

Let D be a set. Note that there exists a family of subsets of D which is empty and there exists a family of subsets of D which is non empty.

In the sequel F, G are families of subsets of D and P is a subset of D.

Let us consider D, F. Then $\bigcup F$ is a subset of D.

Let us consider D, F. Then $\bigcap F$ is a subset of D.

We now state the proposition

(44)[6] If for every P holds $P \in F$ iff $P \in G$, then $F = G$.

The scheme *SubFamEx* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let us consider D, F. The functor F^{c} yielding a family of subsets of D is defined as follows:

(Def. 8) For every subset P of D holds $P \in F^{\mathrm{c}}$ iff $P^{\mathrm{c}} \in F$.

Let us note that the functor F^{c} is involutive.

The following three propositions are true:

(46)[7] If $F \neq \emptyset$, then $F^{\mathrm{c}} \neq \emptyset$.

(47) If $F \neq \emptyset$, then $\Omega_D \setminus \bigcup F = \bigcap (F^{\mathrm{c}})$.

(48) If $F \neq \emptyset$, then $\bigcup (F^{\mathrm{c}}) = \Omega_D \setminus \bigcap F$.

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[3] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

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[6] The propositions (42) and (43) have been removed.

[7] The proposition (45) has been removed.