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Series[1]

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Summary. The article contains definitions and properties of convergentserieses.

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The articles [2], [11], [4], [1], [8], [7], [5], [3], [6], [12], [9], and [10] provide the notation and terminology for this paper.

We follow the rules: n, m are natural numbers, a, p, r are real numbers, and s, s_1, s_2 are sequences of real numbers.

One can prove the following three propositions:

(1) If $0 < a$ and $a < 1$ and for every n holds $s(n) = a^{n+1}$, then s is convergent and $\lim s = 0$.

(2) If $a \neq 0$, then $|a|^n = |a^n|$.

(3) If $|a| < 1$ and for every n holds $s(n) = a^{n+1}$, then s is convergent and $\lim s = 0$.

Let us consider s. The functor $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ yielding a sequence of real numbers is defined as follows:

(Def. 1) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(0) = s(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) + s(n+1)$.

Let us consider s. We say that s is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

The functor $\sum s$ yielding a real number is defined by:

(Def. 3) $\sum s = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})$.

One can prove the following propositions:

(7)[1] If s is summable, then s is convergent and $\lim s = 0$.

(8) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1+s_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(9) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1-s_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(10) If s_1 is summable and s_2 is summable, then s_1+s_2 is summable and $\sum(s_1+s_2) = \sum s_1 + \sum s_2$.

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[1] The propositions (4)–(6) have been removed.

(11) If s_1 is summable and s_2 is summable, then $s_1 - s_2$ is summable and $\sum(s_1 - s_2) = \sum s_1 - \sum s_2$.

(12) $(\sum_{\alpha=0}^{\kappa}(r\,s)(\alpha))_{\kappa\in\mathbb{N}} = r\,(\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}$.

(13) If s is summable, then $r\,s$ is summable and $\sum(r\,s) = r\cdot\sum s$.

(14) For all s, s_1 such that for every n holds $s_1(n) = s(0)$ holds $(\sum_{\alpha=0}^{\kappa}(s\uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}\uparrow 1 - s_1$.

(15) If s is summable, then for every n holds $s\uparrow n$ is summable.

(16) If there exists n such that $s\uparrow n$ is summable, then s is summable.

(17) If for every n holds $s_1(n) \leq s_2(n)$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(18) If s is summable, then for every n holds $\sum s = (\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}(n) + \sum(s\uparrow(n+1))$.

(19) If for every n holds $0 \leq s(n)$, then $(\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(20) If for every n holds $0 \leq s(n)$, then $(\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded iff s is summable.

(21) If s is summable and for every n holds $0 \leq s(n)$, then $0 \leq \sum s$.

(22) If for every n holds $0 \leq s_2(n)$ and s_1 is summable and there exists m such that for every n such that $m \leq n$ holds $s_2(n) \leq s_1(n)$, then s_2 is summable.

(24)[2] If for every n holds $0 \leq s_1(n)$ and $s_1(n) \leq s_2(n)$ and s_2 is summable, then s_1 is summable and $\sum s_1 \leq \sum s_2$.

(25) s is summable iff for every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}(n)| < r$.

(26) If $a \neq 1$, then $(\sum_{\alpha=0}^{\kappa}((a^{\kappa})_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1-a^{n+1}}{1-a}$.

(27) If $a \neq 1$ and for every n holds $s(n+1) = a\cdot s(n)$, then for every n holds $(\sum_{\alpha=0}^{\kappa}s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{s(0)\cdot(1-a^{n+1})}{1-a}$.

(28) If $|a| < 1$, then $(a^{\kappa})_{\kappa\in\mathbb{N}}$ is summable and $\sum((a^{\kappa})_{\kappa\in\mathbb{N}}) = \frac{1}{1-a}$.

(29) If $|a| < 1$ and for every n holds $s(n+1) = a\cdot s(n)$, then s is summable and $\sum s = \frac{s(0)}{1-a}$.

(30) If for every n holds $s(n) > 0$ and $s_1(n) = \frac{s(n+1)}{s(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is summable.

(31) If for every n holds $s(n) > 0$ and there exists m such that for every n such that $n \geq m$ holds $\frac{s(n+1)}{s(n)} \geq 1$, then s is not summable.

(32) If for every n holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is summable.

(33) If for every n holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and there exists m such that for every n such that $m \leq n$ holds $s_1(n) \geq 1$, then s is not summable.

(34) If for every n holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and s_1 is convergent and $\lim s_1 > 1$, then s is not summable.

Let k, n be natural numbers. Then k^n is a natural number.
One can prove the following three propositions:

[2] The proposition (23) has been removed.

(35) Suppose s is non-increasing and for every n holds $s(n) \geq 0$ and $s_1(n) = 2^n \cdot s(2^n)$. Then s is summable if and only if s_1 is summable.

(36) If $p > 1$ and for every n such that $n \geq 1$ holds $s(n) = \frac{1}{n^p}$, then s is summable.

(37) If $p \leq 1$ and for every n such that $n \geq 1$ holds $s(n) = \frac{1}{n^p}$, then s is not summable.

Let us consider s. We say that s is absolutely summable if and only if:

(Def. 5)[3] $|s|$ is summable.

The following propositions are true:

(39)[4] For all n, m such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)| \leq |(\sum_{\alpha=0}^{\kappa} |s|(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} |s|(\alpha))_{\kappa \in \mathbb{N}}(n)|$.

(40) If s is absolutely summable, then s is summable.

(41) If for every n holds $0 \leq s(n)$ and s is summable, then s is absolutely summable.

(42) If for every n holds $s(n) \neq 0$ and $s_1(n) = \frac{|s|(n+1)}{|s|(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is absolutely summable.

(43) If $r > 0$ and there exists m such that for every n such that $n \geq m$ holds $|s(n)| \geq r$, then s is not convergent or $\lim s \neq 0$.

(44) If for every n holds $s(n) \neq 0$ and there exists m such that for every n such that $n \geq m$ holds $\frac{|s|(n+1)}{|s|(n)} \geq 1$, then s is not summable.

(45) If for every n holds $s_1(n) = \sqrt[n]{|s|(n)}$ and s_1 is convergent and $\lim s_1 < 1$, then s is absolutely summable.

(46) If for every n holds $s_1(n) = \sqrt[n]{|s|(n)}$ and there exists m such that for every n such that $m \leq n$ holds $s_1(n) \geq 1$, then s is not summable.

(47) If for every n holds $s_1(n) = \sqrt[n]{|s|(n)}$ and s_1 is convergent and $\lim s_1 > 1$, then s is not summable.

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[3] The definition (Def. 4) has been removed.
[4] The proposition (38) has been removed.

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