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Monotone Real Sequences. Subsequences

Jarosław Kotowicz
Warsaw University
Białystok

Summary. The article contains definitions of constant, increasing, decreasing, non decreasing, non increasing sequences, the definition of a subsequence and their basic properties.

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The articles [7], [2], [8], [9], [3], [5], [4], [1], and [6] provide the notation and terminology for this paper.

We follow the rules: n, m, k are natural numbers, r is a real number, and f, s_1, s_2, s_3 are sequences of real numbers.

Let f be a partial function from $\mathbb{N}$ to $\mathbb{R}$. We say that f is increasing if and only if:

(Def. 1) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) < f(n)$.

We say that f is decreasing if and only if:

(Def. 2) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) > f(n)$.

We say that f is non-decreasing if and only if:

(Def. 3) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) \leq f(n)$.

We say that f is non-increasing if and only if:

(Def. 4) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) \geq f(n)$.

Let f be a function. We say that f is constant if and only if:

(Def. 5) For all sets n_1, n_2 such that $n_1 \in \operatorname{dom} f$ and $n_2 \in \operatorname{dom} f$ holds $f(n_1) = f(n_2)$.

Let us consider s_1. Let us observe that s_1 is constant if and only if:

(Def. 6) There exists r such that for every n holds $s_1(n) = r$.

Let us consider s_1. We say that s_1 is monotone if and only if:

(Def. 7) s_1 is non-decreasing and non-increasing.

The following propositions are true:

(7)[1] s_1 is increasing iff for all n, m such that $n < m$ holds $s_1(n) < s_1(m)$.

[1] The propositions (1)–(6) have been removed.

(8) s_1 is increasing iff for all n, k holds $s_1(n) < s_1(n+1+k)$.

(9) s_1 is decreasing iff for all n, k holds $s_1(n+1+k) < s_1(n)$.

(10) s_1 is decreasing iff for all n, m such that $n < m$ holds $s_1(m) < s_1(n)$.

(11) s_1 is non-decreasing iff for all n, k holds $s_1(n) \leq s_1(n+k)$.

(12) s_1 is non-decreasing iff for all n, m such that $n \leq m$ holds $s_1(n) \leq s_1(m)$.

(13) s_1 is non-increasing iff for all n, k holds $s_1(n+k) \leq s_1(n)$.

(14) s_1 is non-increasing iff for all n, m such that $n \leq m$ holds $s_1(m) \leq s_1(n)$.

(15) s_1 is constant iff there exists r such that $\mathrm{rng}\, s_1 = \{r\}$.

(16) s_1 is constant iff for every n holds $s_1(n) = s_1(n+1)$.

(17) s_1 is constant iff for all n, k holds $s_1(n) = s_1(n+k)$.

(18) s_1 is constant iff for all n, m holds $s_1(n) = s_1(m)$.

(19) If s_1 is increasing, then for every n such that $0 < n$ holds $s_1(0) < s_1(n)$.

(20) If s_1 is decreasing, then for every n such that $0 < n$ holds $s_1(n) < s_1(0)$.

(21) If s_1 is non-decreasing, then for every n holds $s_1(0) \leq s_1(n)$.

(22) If s_1 is non-increasing, then for every n holds $s_1(n) \leq s_1(0)$.

(23) If s_1 is increasing, then s_1 is non-decreasing.

(24) If s_1 is decreasing, then s_1 is non-increasing.

(25) If s_1 is constant, then s_1 is non-decreasing.

(26) If s_1 is constant, then s_1 is non-increasing.

(27) If s_1 is non-decreasing and non-increasing, then s_1 is constant.

Let I_1 be a binary relation. We say that I_1 is natural-yielding if and only if:

(Def. 8) $\mathrm{rng}\, I_1 \subseteq \mathbb{N}$.

Let us note that there exists a sequence of real numbers which is increasing and natural-yielding.
A sequence of naturals is a natural-yielding sequence of real numbers.
Let us consider s_1, k. The functor $s_1 \uparrow k$ yields a sequence of real numbers and is defined by:

(Def. 9) For every n holds $(s_1 \uparrow k)(n) = s_1(n+k)$.

In the sequel N_1, N_2 denote increasing sequences of naturals.
The following propositions are true:

(29)[2] s_1 is an increasing sequence of naturals if and only if s_1 is increasing and for every n holds $s_1(n)$ is a natural number.

(31)[3] For every n holds $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.

Let us consider N_1, n. Then $N_1(n)$ is a natural number.
Let us consider N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of real numbers.
Let us consider N_1, N_2. Then $N_2 \cdot N_1$ is an increasing sequence of naturals.
Let us consider N_1, k. Observe that $N_1 \uparrow k$ is increasing and natural-yielding.
Let us consider s_1, s_2. We say that s_1 is a subsequence of s_2 if and only if:

[2] The proposition (28) has been removed.
[3] The proposition (30) has been removed.

(Def. 10) There exists N_1 such that $s_1 = s_2 \cdot N_1$.

Let f be a sequence of real numbers. Let us observe that f is increasing if and only if:

(Def. 11) For every natural number n holds $f(n) < f(n+1)$.

Let us observe that f is decreasing if and only if:

(Def. 12) For every natural number n holds $f(n) > f(n+1)$.

Let us observe that f is non-decreasing if and only if:

(Def. 13) For every natural number n holds $f(n) \leq f(n+1)$.

Let us observe that f is non-increasing if and only if:

(Def. 14) For every natural number n holds $f(n) \geq f(n+1)$.

We now state a number of propositions:

(33)[4] For every n holds $n \leq N_1(n)$.

(34) $s_1 \uparrow 0 = s_1$.

(35) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(36) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k+m)$.

(37) $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$.

(38) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(39) $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$.

(40) If s_1 is non-zero, then $s_1 \uparrow k$ is non-zero.

(41) $s_1{}^{-1} \uparrow k = (s_1 \uparrow k)^{-1}$.

(42) $(s_1\, s_2) \uparrow k = (s_1 \uparrow k)\,(s_2 \uparrow k)$.

(43) $(s_1/s_2) \uparrow k = (s_1 \uparrow k)/(s_2 \uparrow k)$.

(44) $(r\, s_1) \uparrow k = r\,(s_1 \uparrow k)$.

(45) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(46) s_1 is a subsequence of s_1.

(47) $s_1 \uparrow k$ is a subsequence of s_1.

(48) If s_1 is a subsequence of s_2 and s_2 is a subsequence of s_3, then s_1 is a subsequence of s_3.

(49) If s_1 is increasing and s_2 is a subsequence of s_1, then s_2 is increasing.

(50) If s_1 is decreasing and s_2 is a subsequence of s_1, then s_2 is decreasing.

(51) If s_1 is non-decreasing and s_2 is a subsequence of s_1, then s_2 is non-decreasing.

(52) If s_1 is non-increasing and s_2 is a subsequence of s_1, then s_2 is non-increasing.

(53) If s_1 is monotone and s_2 is a subsequence of s_1, then s_2 is monotone.

(54) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(55) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

[4] The proposition (32) has been removed.

(56) If s_1 is upper bounded and s_2 is a subsequence of s_1, then s_2 is upper bounded.

(57) If s_1 is lower bounded and s_2 is a subsequence of s_1, then s_2 is lower bounded.

(58) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(59)(i) If s_1 is increasing and $0 < r$, then $r\,s_1$ is increasing,
(ii) if $0 = r$, then $r\,s_1$ is constant, and
(iii) if s_1 is increasing and $r < 0$, then $r\,s_1$ is decreasing.

(60) If s_1 is decreasing and $0 < r$, then $r\,s_1$ is decreasing and if s_1 is decreasing and $r < 0$, then $r\,s_1$ is increasing.

(61)(i) If s_1 is non-decreasing and $0 \leq r$, then $r\,s_1$ is non-decreasing, and
(ii) if s_1 is non-decreasing and $r \leq 0$, then $r\,s_1$ is non-increasing.

(62)(i) If s_1 is non-increasing and $0 \leq r$, then $r\,s_1$ is non-increasing, and
(ii) if s_1 is non-increasing and $r \leq 0$, then $r\,s_1$ is non-decreasing.

(63)(i) If s_1 is increasing and s_2 is increasing, then $s_1 + s_2$ is increasing,
(ii) if s_1 is decreasing and s_2 is decreasing, then $s_1 + s_2$ is decreasing,
(iii) if s_1 is non-decreasing and s_2 is non-decreasing, then $s_1 + s_2$ is non-decreasing, and
(iv) if s_1 is non-increasing and s_2 is non-increasing, then $s_1 + s_2$ is non-increasing.

(64)(i) If s_1 is increasing and s_2 is constant, then $s_1 + s_2$ is increasing,
(ii) if s_1 is decreasing and s_2 is constant, then $s_1 + s_2$ is decreasing,
(iii) if s_1 is non-decreasing and s_2 is constant, then $s_1 + s_2$ is non-decreasing, and
(iv) if s_1 is non-increasing and s_2 is constant, then $s_1 + s_2$ is non-increasing.

(65) If s_1 is constant, then for every r holds $r\,s_1$ is constant and $-s_1$ is constant and $|s_1|$ is constant.

(66) If s_1 is constant and s_2 is constant, then $s_1\,s_2$ is constant and $s_1 + s_2$ is constant.

(67) If s_1 is constant and s_2 is constant, then $s_1 - s_2$ is constant.

(68)(i) If s_1 is upper bounded and $0 < r$, then $r\,s_1$ is upper bounded,
(ii) if $0 = r$, then $r\,s_1$ is bounded, and
(iii) if s_1 is upper bounded and $r < 0$, then $r\,s_1$ is lower bounded.

(69)(i) If s_1 is lower bounded and $0 < r$, then $r\,s_1$ is lower bounded, and
(ii) if s_1 is lower bounded and $r < 0$, then $r\,s_1$ is upper bounded.

(70) If s_1 is bounded, then for every r holds $r\,s_1$ is bounded and $-s_1$ is bounded and $|s_1|$ is bounded.

(71)(i) If s_1 is upper bounded and s_2 is upper bounded, then $s_1 + s_2$ is upper bounded,
(ii) if s_1 is lower bounded and s_2 is lower bounded, then $s_1 + s_2$ is lower bounded, and
(iii) if s_1 is bounded and s_2 is bounded, then $s_1 + s_2$ is bounded.

(72) If s_1 is bounded and s_2 is bounded, then $s_1\,s_2$ is bounded and $s_1 - s_2$ is bounded.

(73) If s_1 is constant, then s_1 is bounded.

(74) If s_1 is constant, then for every r holds $r\,s_1$ is bounded and $-s_1$ is bounded and $|s_1|$ is bounded.

(75)(i) If s_1 is upper bounded and s_2 is constant, then $s_1 + s_2$ is upper bounded,
(ii) if s_1 is lower bounded and s_2 is constant, then $s_1 + s_2$ is lower bounded, and
(iii) if s_1 is bounded and s_2 is constant, then $s_1 + s_2$ is bounded.

(76)(i) If s_1 is upper bounded and s_2 is constant, then $s_1 - s_2$ is upper bounded,
(ii) if s_1 is lower bounded and s_2 is constant, then $s_1 - s_2$ is lower bounded, and
(iii) if s_1 is bounded and s_2 is constant, then $s_1 - s_2$ is bounded and $s_2 - s_1$ is bounded and $s_1\ s_2$ is bounded.

(77) If s_1 is upper bounded and s_2 is non-increasing, then $s_1 + s_2$ is upper bounded.

(78) If s_1 is lower bounded and s_2 is non-decreasing, then $s_1 + s_2$ is lower bounded.

(79) For all sets X, x holds $X \longmapsto x$ is constant.

Let X, x be sets. Observe that $X \longmapsto x$ is constant.

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