JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Convergent Real Sequences. Upper and Lower Bound of Sets of Real Numbers

Jarosław Kotowicz
Warsaw University
Białystok

Summary. The article contains theorems about convergent sequences and the limit of sequences occurring in [5] such as Bolzano-Weierstrass theorem, Cauchy theorem and others. Bounded sets of real numbers and lower and upper bound of subset of real numbers are defined.

MML Identifier: `SEQ_4`.

WWW: `http://mizar.org/JFM/Vol1/seq_4.html`

The articles [9], [12], [2], [11], [4], [13], [7], [5], [1], [3], [6], [8], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: n, k, m denote natural numbers, r, r_1, p, g, g_1, g_2, s denote real numbers, s_1, s_2 denote sequences of real numbers, N_1 denotes an increasing sequence of naturals, and X, Y denote subsets of $\mathbb{R}$.

Next we state several propositions:

(1) If $0 < r_1$ and $r_1 \leq r$ and $0 \leq g$, then $\frac{g}{r} \leq \frac{g}{r_1}$.

(4)[1] If $0 < s$, then $0 < \frac{s}{3}$.

(6)[2] If $0 < g$ and $0 \leq r$ and $g \leq g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$.

(7) If $0 \leq g$ and $0 \leq r$ and $g \leq g_1$ and $r \leq r_1$, then $g \cdot r \leq g_1 \cdot r_1$.

(8) Let given X, Y. Suppose that for all r, p such that $r \in X$ and $p \in Y$ holds $r < p$. Then there exists g such that for all r, p such that $r \in X$ and $p \in Y$ holds $r \leq g$ and $g \leq p$.

(9) If $0 < p$ and there exists r such that $r \in X$ and for every r such that $r \in X$ holds $r + p \in X$, then for every g there exists r such that $r \in X$ and $g < r$.

(10) For every r there exists n such that $r < n$.

Let X be a real-membered set. We say that X is upper bounded if and only if:

(Def. 1) There exists p such that for every r such that $r \in X$ holds $r \leq p$.

We say that X is lower bounded if and only if:

(Def. 2) There exists p such that for every r such that $r \in X$ holds $p \leq r$.

[1] The propositions (2) and (3) have been removed.
[2] The proposition (5) has been removed.

© Association of Mizar Users

Let us consider X. We say that X is bounded if and only if:

(Def. 3) X is lower bounded and upper bounded.

One can prove the following proposition

(14)[3] X is bounded iff there exists s such that $0 < s$ and for every r such that $r \in X$ holds $|r| < s$.

Let us consider r. Then $\{r\}$ is a subset of $\mathbb{R}$.
The following propositions are true:

(15) $\{r\}$ is bounded.

(16) Let X be a real-membered set. Suppose X is non empty and upper bounded. Then there exists g such that for every r such that $r \in X$ holds $r \leq g$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $g - s < r$.

(17) Let X be a real-membered set. Suppose that

(i) for every r such that $r \in X$ holds $r \leq g_1$,

(ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_1 - s < r$,

(iii) for every r such that $r \in X$ holds $r \leq g_2$, and

(iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_2 - s < r$.

Then $g_1 = g_2$.

(18) Let X be a real-membered set. Suppose X is non empty and lower bounded. Then there exists g such that for every r such that $r \in X$ holds $g \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g + s$.

(19) Let X be a real-membered set. Suppose that

(i) for every r such that $r \in X$ holds $g_1 \leq r$,

(ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_1 + s$,

(iii) for every r such that $r \in X$ holds $g_2 \leq r$, and

(iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_2 + s$.

Then $g_1 = g_2$.

Let X be a real-membered set. Let us assume that X is non empty and upper bounded. The functor $\sup X$ yields a real number and is defined as follows:

(Def. 4) For every r such that $r \in X$ holds $r \leq \sup X$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $\sup X - s < r$.

Let X be a real-membered set. Let us assume that X is non empty and lower bounded. The functor $\inf X$ yielding a real number is defined by:

(Def. 5) For every r such that $r \in X$ holds $\inf X \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < \inf X + s$.

Let us consider X. Then $\sup X$ is a real number. Then $\inf X$ is a real number.
One can prove the following propositions:

(22)[4] $\inf\{r\} = r$ and $\sup\{r\} = r$.

(23) $\inf\{r\} = \sup\{r\}$.

(24) If X is bounded and non empty, then $\inf X \leq \sup X$.

[3] The propositions (11)–(13) have been removed.
[4] The propositions (20) and (21) have been removed.

(25) If X is bounded and non empty, then there exist r, p such that $r \in X$ and $p \in X$ and $p \neq r$ iff $\inf X < \sup X$.

(26) If s_1 is convergent, then $|s_1|$ is convergent.

(27) If s_1 is convergent, then $\lim|s_1| = |\lim s_1|$.

(28) If $|s_1|$ is convergent and $\lim|s_1| = 0$, then s_1 is convergent and $\lim s_1 = 0$.

(29) If s_2 is a subsequence of s_1 and s_1 is convergent, then s_2 is convergent.

(30) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(31) If s_1 is convergent and there exists k such that for every n such that $k \leq n$ holds $s_2(n) = s_1(n)$, then s_2 is convergent.

(32) If s_1 is convergent and there exists k such that for every n such that $k \leq n$ holds $s_2(n) = s_1(n)$, then $\lim s_1 = \lim s_2$.

(33) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(35)[5] If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(36) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then $\lim s_2 = \lim s_1$.

(37) If s_1 is convergent and $\lim s_1 \neq 0$, then there exists k such that $s_1 \uparrow k$ is non-zero.

(38) If s_1 is convergent and $\lim s_1 \neq 0$, then there exists s_2 which is a subsequence of s_1 and non-zero.

(39) If s_1 is constant, then s_1 is convergent.

(40) If s_1 is constant and $r \in \operatorname{rng} s_1$ or s_1 is constant and there exists n such that $s_1(n) = r$, then $\lim s_1 = r$.

(41) If s_1 is constant, then for every n holds $\lim s_1 = s_1(n)$.

(42) If s_1 is convergent and $\lim s_1 \neq 0$, then for every s_2 such that s_2 is a subsequence of s_1 and non-zero holds $\lim(s_2{}^{-1}) = (\lim s_1)^{-1}$.

(43) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then s_1 is convergent.

(44) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then $\lim s_1 = 0$.

(45) If for every n holds $s_1(n) = \frac{1}{n+1}$, then s_1 is convergent and $\lim s_1 = 0$.

(46) If $0 < r$ and for every n holds $s_1(n) = \frac{g}{n+r}$, then s_1 is convergent and $\lim s_1 = 0$.

(47) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n \cdot n+r}$, then s_1 is convergent.

(48) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n \cdot n+r}$, then $\lim s_1 = 0$.

(49) If for every n holds $s_1(n) = \frac{1}{n \cdot n+1}$, then s_1 is convergent and $\lim s_1 = 0$.

(50) If $0 < r$ and for every n holds $s_1(n) = \frac{g}{n \cdot n+r}$, then s_1 is convergent and $\lim s_1 = 0$.

(51) If s_1 is non-decreasing and upper bounded, then s_1 is convergent.

(52) If s_1 is non-increasing and lower bounded, then s_1 is convergent.

(53) If s_1 is monotone and bounded, then s_1 is convergent.

(54) If s_1 is upper bounded and non-decreasing, then for every n holds $s_1(n) \leq \lim s_1$.

[5] The proposition (34) has been removed.

(55) If s_1 is lower bounded and non-increasing, then for every n holds $\lim s_1 \leq s_1(n)$.

(56) For every s_1 there exists N_1 such that $s_1 \cdot N_1$ is monotone.

(57) If s_1 is bounded, then there exists s_2 which is a subsequence of s_1 and convergent.

(58) s_1 is convergent iff for every s such that $0 < s$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - s_1(n)| < s$.

(59) If s_1 is constant and s_2 is convergent, then $\lim(s_1 + s_2) = s_1(0) + \lim s_2$ and $\lim(s_1 - s_2) = s_1(0) - \lim s_2$ and $\lim(s_2 - s_1) = \lim s_2 - s_1(0)$ and $\lim(s_1\ s_2) = s_1(0) \cdot \lim s_2$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[5] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_2.html.

[6] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seqm_3.html.

[7] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[8] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[10] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol16/membered.html.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received November 23, 1989

Published January 2, 2004