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# Subspaces and Cosets of Subspace of Real Unitary Space

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**Summary.** In this article, subspace and the coset of subspace of real unitary space are defined. And we discuss some of their fundamental properties.

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The articles [6], [3], [11], [8], [7], [1], [12], [2], [5], [10], [9], and [4] provide the notation and terminology for this paper.

## 1. Definition and Axioms of the Subspace of Real Unitary Space

Let $V$ be a real unitary space. A real unitary space is said to be a subspace of $V$ if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of $V$,

(ii) the zero of it $=$ the zero of $V$,

(iii) the addition of it $=$ (the addition of $V$)$\restriction$[: the carrier of it, the carrier of it :],

(iv) the external multiplication of it $=$ (the external multiplication of $V$)$\restriction$[: $\mathbb{R}$, the carrier of it :], and

(v) the scalar product of it $=$ (the scalar product of $V$)$\restriction$[: the carrier of it, the carrier of it :].

We now state a number of propositions:

(1) Let $V$ be a real unitary space, $W_1$, $W_2$ be subspaces of $V$, and $x$ be a set. If $x \in W_1$ and $W_1$ is a subspace of $W_2$, then $x \in W_2$.

(2) For every real unitary space $V$ and for every subspace $W$ of $V$ and for every set $x$ such that $x \in W$ holds $x \in V$.

(3) For every real unitary space $V$ and for every subspace $W$ of $V$ holds every vector of $W$ is a vector of $V$.

(4) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $0_W = 0_V$.

(5) For every real unitary space $V$ and for all subspaces $W_1$, $W_2$ of $V$ holds $0_{(W_1)} = 0_{(W_2)}$.

(6) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $u$, $v$ be vectors of $V$, and $w_1$, $w_2$ be vectors of $W$. If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(7) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, $w$ be a vector of $W$, and $a$ be a real number. If $w = v$, then $a \cdot w = a \cdot v$.

(8) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v_1$, $v_2$ be vectors of $V$, and $w_1$, $w_2$ be vectors of $W$. If $w_1 = v_1$ and $w_2 = v_2$, then $(w_1|w_2) = (v_1|v_2)$.

(9) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, and $w$ be a vector of $W$. If $w = v$, then $-v = -w$.

(10) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $u$, $v$ be vectors of $V$, and $w_1$, $w_2$ be vectors of $W$. If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(11) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $0_V \in W$.

(12) For every real unitary space $V$ and for all subspaces $W_1$, $W_2$ of $V$ holds $0_{(W_1)} \in W_2$.

(13) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $0_W \in V$.

(14) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. If $u \in W$ and $v \in W$, then $u + v \in W$.

(15) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, and $a$ be a real number. If $v \in W$, then $a \cdot v \in W$.

(16) For every real unitary space $V$ and for every subspace $W$ of $V$ and for every vector $v$ of $V$ such that $v \in W$ holds $-v \in W$.

(17) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. If $u \in W$ and $v \in W$, then $u - v \in W$.

(18) Let $V$ be a real unitary space, $V_1$ be a subset of $V$, $D$ be a non empty set, $d_1$ be an element of $D$, $A$ be a binary operation on $D$, $M$ be a function from $[:\mathbb{R}, D:]$ into $D$, and $S$ be a function from $[:D, D:]$ into $\mathbb{R}$. Suppose that

(i) $V_1 = D$,

(ii) $d_1 = 0_V$,

(iii) $A = (\text{the addition of } V) \restriction [:V_1, V_1:]$,

(iv) $M = (\text{the external multiplication of } V) \restriction [:\mathbb{R}, V_1:]$, and

(v) $S = (\text{the scalar product of } V) \restriction [:V_1, V_1:]$.

Then $\langle D, d_1, A, M, S \rangle$ is a subspace of $V$.

(19) Every real unitary space $V$ is a subspace of $V$.

(20) For all strict real unitary spaces $V$, $X$ such that $V$ is a subspace of $X$ and $X$ is a subspace of $V$ holds $V = X$.

(21) Let $V$, $X$, $Y$ be real unitary spaces. Suppose $V$ is a subspace of $X$ and $X$ is a subspace of $Y$. Then $V$ is a subspace of $Y$.

(22) Let $V$ be a real unitary space and $W_1$, $W_2$ be subspaces of $V$. Suppose the carrier of $W_1 \subseteq$ the carrier of $W_2$. Then $W_1$ is a subspace of $W_2$.

(23) Let $V$ be a real unitary space and $W_1$, $W_2$ be subspaces of $V$. Suppose that for every vector $v$ of $V$ such that $v \in W_1$ holds $v \in W_2$. Then $W_1$ is a subspace of $W_2$.

Let $V$ be a real unitary space. Note that there exists a subspace of $V$ which is strict.

Next we state several propositions:

(24) Let $V$ be a real unitary space and $W_1$, $W_2$ be strict subspaces of $V$. If the carrier of $W_1$ = the carrier of $W_2$, then $W_1 = W_2$.

(25) Let $V$ be a real unitary space and $W_1$, $W_2$ be strict subspaces of $V$. If for every vector $v$ of $V$ holds $v \in W_1$ iff $v \in W_2$, then $W_1 = W_2$.

(26) Let $V$ be a strict real unitary space and $W$ be a strict subspace of $V$. If the carrier of $W$ = the carrier of $V$, then $W = V$.

(27) Let $V$ be a strict real unitary space and $W$ be a strict subspace of $V$. If for every vector $v$ of $V$ holds $v \in W$ iff $v \in V$, then $W = V$.

(28) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $V_1$ be a subset of $V$. If the carrier of $W = V_1$, then $V_1$ is linearly closed.

(29) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $V_1$ be a subset of $V$. Suppose $V_1 \neq \emptyset$ and $V_1$ is linearly closed. Then there exists a strict subspace $W$ of $V$ such that $V_1$ = the carrier of $W$.

## 2. Definition of Zero Subspace and Improper Subspace of Real Unitary Space

Let $V$ be a real unitary space. The functor $\mathbf{0}_V$ yields a strict subspace of $V$ and is defined as follows:

(Def. 2) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let $V$ be a real unitary space. The functor $\Omega_V$ yielding a strict subspace of $V$ is defined as follows:

(Def. 3) $\Omega_V$ = the unitary space structure of $V$.

## 3. Theorems of Zero Subspace and Improper Subspace

The following propositions are true:

(30) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $\mathbf{0}_W = \mathbf{0}_V$.

(31) For every real unitary space $V$ and for all subspaces $W_1$, $W_2$ of $V$ holds $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(32) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $\mathbf{0}_W$ is a subspace of $V$.

(33) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $\mathbf{0}_V$ is a subspace of $W$.

(34) For every real unitary space $V$ and for all subspaces $W_1$, $W_2$ of $V$ holds $\mathbf{0}_{(W_1)}$ is a subspace of $W_2$.

(35) Every strict real unitary space $V$ is a subspace of $\Omega_V$.

## 4. The Cosets of Subspace of Real Unitary Space

Let $V$ be a real unitary space, let $v$ be a vector of $V$, and let $W$ be a subspace of $V$. The functor $v+W$ yielding a subset of $V$ is defined by:

(Def. 4) $v+W = \{v+u; u$ ranges over vectors of $V$: $u \in W\}$.

Let $V$ be a real unitary space and let $W$ be a subspace of $V$. A subset of $V$ is called a coset of $W$ if:

(Def. 5) There exists a vector $v$ of $V$ such that it $= v+W$.

## 5. Theorems of the Cosets

One can prove the following propositions:

(36) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v$ be a vector of $V$. Then $0_V \in v+W$ if and only if $v \in W$.

(37) For every real unitary space $V$ and for every subspace $W$ of $V$ and for every vector $v$ of $V$ holds $v \in v+W$.

(38) For every real unitary space $V$ and for every subspace $W$ of $V$ holds $0_V + W =$ the carrier of $W$.

(39) For every real unitary space $V$ and for every vector $v$ of $V$ holds $v + \mathbf{0}_V = \{v\}$.

(40) For every real unitary space $V$ and for every vector $v$ of $V$ holds $v + \Omega_V =$ the carrier of $V$.

(41) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v$ be a vector of $V$. Then $0_V \in v+W$ if and only if $v+W =$ the carrier of $W$.

(42) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v$ be a vector of $V$. Then $v \in W$ if and only if $v+W =$ the carrier of $W$.

(43) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, and $a$ be a real number. If $v \in W$, then $a \cdot v + W =$ the carrier of $W$.

(44) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, and $a$ be a real number. If $a \neq 0$ and $a \cdot v + W =$ the carrier of $W$, then $v \in W$.

(45) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v$ be a vector of $V$. Then $v \in W$ if and only if $-v+W =$ the carrier of $W$.

(46) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $u \in W$ if and only if $v+W = v+u+W$.

(47) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $u \in W$ if and only if $v+W = (v-u)+W$.

(48) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $v \in u+W$ if and only if $u+W = v+W$.

(49) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v$ be a vector of $V$. Then $v+W = -v+W$ if and only if $v \in W$.

(50) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v_1$, $v_2$ be vectors of $V$. If $u \in v_1+W$ and $u \in v_2+W$, then $v_1+W = v_2+W$.

(51) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. If $u \in v+W$ and $u \in -v+W$, then $v \in W$.

(52) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, and $a$ be a real number. If $a \neq 1$ and $a \cdot v \in v+W$, then $v \in W$.

(53) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $v$ be a vector of $V$, and $a$ be a real number. If $v \in W$, then $a \cdot v \in v+W$.

(54) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v$ be a vector of $V$. Then $-v \in v+W$ if and only if $v \in W$.

(55) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $u+v \in v+W$ if and only if $u \in W$.

(56) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $v-u \in v+W$ if and only if $u \in W$.

(57) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $u \in v+W$ if and only if there exists a vector $v_1$ of $V$ such that $v_1 \in W$ and $u = v+v_1$.

(58) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. Then $u \in v+W$ if and only if there exists a vector $v_1$ of $V$ such that $v_1 \in W$ and $u = v-v_1$.

(59) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v_1$, $v_2$ be vectors of $V$. Then there exists a vector $v$ of $V$ such that $v_1 \in v+W$ and $v_2 \in v+W$ if and only if $v_1-v_2 \in W$.

(60) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. If $v+W = u+W$, then there exists a vector $v_1$ of $V$ such that $v_1 \in W$ and $v+v_1 = u$.

(61) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $u$, $v$ be vectors of $V$. If $v+W = u+W$, then there exists a vector $v_1$ of $V$ such that $v_1 \in W$ and $v-v_1 = u$.

(62) Let $V$ be a real unitary space, $W_1$, $W_2$ be strict subspaces of $V$, and $v$ be a vector of $V$. Then $v+W_1 = v+W_2$ if and only if $W_1 = W_2$.

(63) Let $V$ be a real unitary space, $W_1$, $W_2$ be strict subspaces of $V$, and $u$, $v$ be vectors of $V$. If $v+W_1 = u+W_2$, then $W_1 = W_2$.

(64) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $C$ be a coset of $W$. Then $C$ is linearly closed if and only if $C =$ the carrier of $W$.

(65) Let $V$ be a real unitary space, $W_1$, $W_2$ be strict subspaces of $V$, $C_1$ be a coset of $W_1$, and $C_2$ be a coset of $W_2$. If $C_1 = C_2$, then $W_1 = W_2$.

(66) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $C$ be a coset of $W$, and $v$ be a vector of $V$. Then $\{v\}$ is a coset of $\mathbf{0}_V$.

(67) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $V_1$ be a subset of $V$, and $v$ be a vector of $V$. If $V_1$ is a coset of $\mathbf{0}_V$, then there exists a vector $v$ of $V$ such that $V_1 = \{v\}$.

(68) For every real unitary space $V$ and for every subspace $W$ of $V$ holds the carrier of $W$ is a coset of $W$.

(69) For every real unitary space $V$ holds the carrier of $V$ is a coset of $\Omega_V$.

(70) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $V_1$ be a subset of $V$. If $V_1$ is a coset of $\Omega_V$, then $V_1 =$ the carrier of $V$.

(71) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $C$ be a coset of $W$. Then $0_V \in C$ if and only if $C =$ the carrier of $W$.

(72) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $C$ be a coset of $W$, and $u$ be a vector of $V$. Then $u \in C$ if and only if $C = u+W$.

(73) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $C$ be a coset of $W$, and $u$, $v$ be vectors of $V$. If $u \in C$ and $v \in C$, then there exists a vector $v_1$ of $V$ such that $v_1 \in W$ and $u+v_1 = v$.

(74) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $C$ be a coset of $W$, and $u$, $v$ be vectors of $V$. If $u \in C$ and $v \in C$, then there exists a vector $v_1$ of $V$ such that $v_1 \in W$ and $u-v_1 = v$.

(75) Let $V$ be a real unitary space, $W$ be a subspace of $V$, and $v_1$, $v_2$ be vectors of $V$. Then there exists a coset $C$ of $W$ such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1-v_2 \in W$.

(76) Let $V$ be a real unitary space, $W$ be a subspace of $V$, $u$ be a vector of $V$, and $B$, $C$ be cosets of $W$. If $u \in B$ and $u \in C$, then $B = C$.

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