JOURNAL OF FORMALIZED MATHEMATICS
Volume 16, Released 2004, Published 2004
Inst. of Computer Science, Univ. of Białystok

Banach Space of Bounded Real Sequences

Yasumasa Suzuki
2-14-18 Take, Yokosuka City
Kanagawa Pref., Japan

Summary. We introduce the arithmetic addition and multiplication in the set of bounded real sequences and introduce the norm also. This set has the structure of the Banach space.

MML Identifier: RSSPACE4.

WWW: http://mizar.org/JFM/Vol16/rsspace4.html

The articles [21], [6], [27], [29], [28], [15], [20], [3], [1], [2], [24], [23], [9], [4], [5], [7], [26], [22], [16], [17], [13], [11], [12], [10], [25], [14], [8], [19], and [18] provide the notation and terminology for this paper.

1. The Banach Space of Bounded Real Sequences

The subset the set of bounded real sequences of the linear space of real sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of bounded real sequences if and only if $x \in$ the set of real sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded.

Let us mention that the set of bounded real sequences is non empty and the set of bounded real sequences is linearly closed.

The following proposition is true

(1) $\langle$the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences)$\rangle$ is a subspace of the linear space of real sequences.

One can verify that $\langle$the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences)$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The function linfty-norm from the set of bounded real sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of bounded real sequences holds linfty-norm$(x) = \sup \mathrm{rng} |\mathrm{id}_{\mathrm{seq}}(x)|$.

We now state the proposition

(2) Let r_1 be a sequence of real numbers. Then r_1 is bounded and $\sup \mathrm{rng}|r_1| = 0$ if and only if for every natural number n holds $r_1(n) = 0$.

Let us note that $\langle$the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences), linfty-norm$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The non empty normed structure linfty-Space is defined by the condition (Def. 3).

(Def. 3) linfty-Space $=$ $\langle$the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences), linfty-norm$\rangle$.

One can prove the following two propositions:

(3) The carrier of linfty-Space $=$ the set of bounded real sequences and for every set x holds x is a vector of linfty-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded and $0_{\text{linfty-Space}} =$ Zeroseq and for every vector u of linfty-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of linfty-Space holds $u+v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$ and for every real number r and for every vector u of linfty-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of linfty-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of linfty-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for every vector v of linfty-Space holds $\mathrm{id}_{\mathrm{seq}}(v)$ is bounded and for every vector v of linfty-Space holds $\|v\| = \sup \mathrm{rng} |\mathrm{id}_{\mathrm{seq}}(v)|$.

(4) Let x, y be points of linfty-Space and a be a real number. Then $\|x\| = 0$ iff $x = 0_{\text{linfty-Space}}$ and $0 \leq \|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

Let us mention that linfty-Space is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following proposition

(5) For every sequence v_1 of linfty-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

2. The Banach Space of Bounded Functions

Let X be a non empty set, let Y be a real normed space, and let I_1 be a function from X into the carrier of Y. We say that I_1 is bounded if and only if:

(Def. 4) There exists a real number K such that $0 \leq K$ and for every element x of X holds $\|I_1(x)\| \leq K$.

We now state the proposition

(6) Let X be a non empty set, Y be a real normed space, and f be a function from X into the carrier of Y. If for every element x of X holds $f(x) = 0_Y$, then f is bounded.

Let X be a non empty set and let Y be a real normed space. Observe that there exists a function from X into the carrier of Y which is bounded.

Let X be a non empty set and let Y be a real normed space. The functor BoundedFunctions(X,Y) yielding a subset of RealVectSpace(X,Y) is defined as follows:

(Def. 5) For every set x holds $x \in$ BoundedFunctions(X,Y) iff x is a bounded function from X into the carrier of Y.

Let X be a non empty set and let Y be a real normed space. One can check that BoundedFunctions(X,Y) is non empty.

We now state two propositions:

(7) For every non empty set X and for every real normed space Y holds BoundedFunctions(X,Y) is linearly closed.

(8) For every non empty set X and for every real normed space Y holds $\langle$BoundedFunctions(X,Y), Zero_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Add_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Mult_(BoundedFunctions(X,Y), RealVectSpace(X,Y))$\rangle$ is a subspace of RealVectSpace(X,Y).

Let X be a non empty set and let Y be a real normed space. One can check that $\langle$BoundedFunctions(X,Y), Zero_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Add_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Mult_(BoundedFunctions(X,Y), RealVectSpace(X,Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(9) For every non empty set X and for every real normed space Y holds $\langle$BoundedFunctions(X,Y), Zero_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Add_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Mult_(BoundedFunctions(X,Y), RealVectSpace(X,Y))$\rangle$ is a real linear space.

Let X be a non empty set and let Y be a real normed space. The set of bounded real sequences from X into Y yields a real linear space and is defined by:

(Def. 6) The set of bounded real sequences from X into $Y = \langle$BoundedFunctions(X,Y), Zero_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Add_(BoundedFunctions(X,Y), RealVectSpace(X,Y)), Mult_(BoundedFunctions(X,Y), RealVectSpace(X,Y))$\rangle$.

Let X be a non empty set and let Y be a real normed space. Note that the set of bounded real sequences from X into Y is strict.

We now state three propositions:

(10) Let X be a non empty set, Y be a real normed space, f, g, h be vectors of the set of bounded real sequences from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(11) Let X be a non empty set, Y be a real normed space, f, h be vectors of the set of bounded real sequences from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h'(x) = a \cdot f'(x)$.

(12) Let X be a non empty set and Y be a real normed space. Then $0_{\text{the set of bounded real sequences from } X \text{ into } Y} = X \longmapsto 0_Y$.

Let X be a non empty set, let Y be a real normed space, and let f be a set. Let us assume that $f \in$ BoundedFunctions(X,Y). The functor modetrans(f,X,Y) yields a bounded function from X into the carrier of Y and is defined by:

(Def. 7) modetrans$(f,X,Y) = f$.

Let X be a non empty set, let Y be a real normed space, and let u be a function from X into the carrier of Y. The functor PreNorms(u) yields a non empty subset of $\mathbb{R}$ and is defined by:

(Def. 8) PreNorms$(u) = \{\|u(t)\| : t$ ranges over elements of $X\}$.

Next we state three propositions:

(13) Let X be a non empty set, Y be a real normed space, and g be a bounded function from X into the carrier of Y. Then PreNorms(g) is non empty and upper bounded.

(14) Let X be a non empty set, Y be a real normed space, and g be a function from X into the carrier of Y. Then g is bounded if and only if PreNorms(g) is upper bounded.

(15) Let X be a non empty set and Y be a real normed space. Then there exists a function N_1 from BoundedFunctions(X,Y) into $\mathbb{R}$ such that for every set f if $f \in$ BoundedFunctions(X,Y), then $N_1(f) = \sup$PreNorms(modetrans(f,X,Y)).

Let X be a non empty set and let Y be a real normed space. The functor BoundedFunctionsNorm(X,Y) yields a function from BoundedFunctions(X,Y) into $\mathbb{R}$ and is defined as follows:

(Def. 9) For every set x such that $x \in$ BoundedFunctions(X,Y) holds BoundedFunctionsNorm$(X,Y)(x) = \sup$PreNorms(modetrans(x,X,Y)).

Next we state two propositions:

(16) Let X be a non empty set, Y be a real normed space, and f be a bounded function from X into the carrier of Y. Then $\mathrm{modetrans}(f,X,Y)=f$.

(17) Let X be a non empty set, Y be a real normed space, and f be a bounded function from X into the carrier of Y. Then $\mathrm{BoundedFunctionsNorm}(X,Y)(f)=\sup \mathrm{PreNorms}(f)$.

Let X be a non empty set and let Y be a real normed space. The real normed space of bounded functions from X into Y yielding a non empty normed structure is defined by:

(Def. 10) The real normed space of bounded functions from X into $Y = \langle \mathrm{BoundedFunctions}(X,Y), \mathrm{Zero\_}(\mathrm{BoundedFunctions}(X,Y), \mathrm{RealVectSpace}(X,Y)), \mathrm{Add\_}(\mathrm{BoundedFunctions}(X,Y), \mathrm{RealVectSpace}(X,Y)), \mathrm{Mult\_}(\mathrm{BoundedFunctions}(X,Y), \mathrm{RealVectSpace}(X,Y)), \mathrm{BoundedFunctionsNorm}(X,Y)\rangle$.

The following propositions are true:

(18) Let X be a non empty set and Y be a real normed space. Then $X \longmapsto 0_Y = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$.

(19) Let X be a non empty set, Y be a real normed space, f be a point of the real normed space of bounded functions from X into Y, and g be a bounded function from X into the carrier of Y. If $g=f$, then for every element t of X holds $\|g(t)\| \leq \|f\|$.

(20) Let X be a non empty set, Y be a real normed space, and f be a point of the real normed space of bounded functions from X into Y. Then $0 \leq \|f\|$.

(21) Let X be a non empty set, Y be a real normed space, and f be a point of the real normed space of bounded functions from X into Y. Suppose $f = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$. Then $0=\|f\|$.

(22) Let X be a non empty set, Y be a real normed space, f, g, h be points of the real normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f'=f$ and $g'=g$ and $h'=h$. Then $h=f+g$ if and only if for every element x of X holds $h'(x)=f'(x)+g'(x)$.

(23) Let X be a non empty set, Y be a real normed space, f, h be points of the real normed space of bounded functions from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f'=f$ and $h'=h$. Let a be a real number. Then $h=a\cdot f$ if and only if for every element x of X holds $h'(x)=a\cdot f'(x)$.

(24) Let X be a non empty set, Y be a real normed space, f, g be points of the real normed space of bounded functions from X into Y, and a be a real number. Then

(i) $\|f\|=0$ iff $f=0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$,

(ii) $\|a\cdot f\|=|a|\cdot\|f\|$, and

(iii) $\|f+g\|\leq\|f\|+\|g\|$.

(25) Let X be a non empty set and Y be a real normed space. Then the real normed space of bounded functions from X into Y is real normed space-like.

(26) Let X be a non empty set and Y be a real normed space. Then the real normed space of bounded functions from X into Y is a real normed space.

Let X be a non empty set and let Y be a real normed space. Observe that the real normed space of bounded functions from X into Y is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

The following three propositions are true:

(27) Let X be a non empty set, Y be a real normed space, f, g, h be points of the real normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f'=f$ and $g'=g$ and $h'=h$. Then $h=f-g$ if and only if for every element x of X holds $h'(x)=f'(x)-g'(x)$.

(28) Let X be a non empty set and Y be a real normed space. Suppose Y is complete. Let s_1 be a sequence of the real normed space of bounded functions from X into Y. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(29) Let X be a non empty set and Y be a real Banach space. Then the real normed space of bounded functions from X into Y is a real Banach space.

Let X be a non empty set and let Y be a real Banach space. Observe that the real normed space of bounded functions from X into Y is complete.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[3] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/pscomp_1.html`.

[8] Noboru Endou, Yasumasa Suzuki, and Yasunari Shidama. Real linear space of real sequences. *Journal of Formalized Mathematics*, 15, 2003. `http://mizar.org/JFM/Vol15/rsspace.html`.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[10] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_4.html`.

[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_2.html`.

[12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seqm_3.html`.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_1.html`.

[14] Jarosław Kotowicz. Properties of real functions. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rfunct_2.html`.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[16] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/absvalue.html`.

[17] Jan Popiołek. Real normed space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/normsp_1.html`.

[18] Yasunari Shidama. Banach space of bounded linear operators. *Journal of Formalized Mathematics*, 15, 2003. `http://mizar.org/JFM/Vol15/lopban_1.html`.

[19] Yasumasa Suzuki, Noboru Endou, and Yasunari Shidama. Banach space of absolute summable real sequences. *Journal of Formalized Mathematics*, 15, 2003. `http://mizar.org/JFM/Vol15/rsspace3.html`.

[20] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funcop_1.html`.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[22] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fraenkel.html`.

[23] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. `http://mizar.org/JFM/Vol16/membered.html`.

[24] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[25] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlsub_1.html`.

[26] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[27] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[28] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[29] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received January 6, 2004

Published January 6, 2004