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Linear Combinations in Right Module over Associative Ring

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The articles [11], [10], [16], [5], [2], [17], [3], [4], [12], [1], [13], [14], [6], [15], [7], [8], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: R denotes a ring, V denotes a right module over R, a, b denote scalars of R, x denotes a set, i, k denote natural numbers, u, v, v_1, v_2, v_3, w denote vectors of V, F, G denote finite sequences of elements of the carrier of V, A, B denote subsets of V, f denotes a function from the carrier of V into the carrier of R, and S, T denote finite subsets of V.

One can prove the following propositions:

(1) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = v \cdot a$, then $\sum F = \sum G \cdot a$.

(2) $\sum(\varepsilon_{(\text{the carrier of } V)}) \cdot a = 0_V$.

(3) $\sum\langle v, u\rangle \cdot a = v \cdot a + u \cdot a$.

(4) $\sum\langle v, u, w\rangle \cdot a = v \cdot a + u \cdot a + w \cdot a$.

Let us consider R, let us consider V, and let us consider T. The functor $\sum T$ yields a vector of V and is defined by:

(Def. 3)[1] There exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

Next we state a number of propositions:

(5) $\sum(\emptyset_V) = 0_V$.

(6) $\sum\{v\} = v$.

(7) If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(8) If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = v_1 + v_2 + v_3$.

(9) If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

(10) $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(11) $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

[1] The definitions (Def. 1) and (Def. 2) have been removed.

(12) $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(13) $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(14) $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(15) $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let us consider R and let us consider V. An element of (the carrier of $R)^{\text{the carrier of } V}$ is said to be a linear combination of V if:

(Def. 4) There exists T such that for every v such that $v \notin T$ holds $\mathrm{it}(v) = 0_R$.

In the sequel L, L_1, L_2, L_3 are linear combinations of V.

Let us consider R, let us consider V, and let us consider L. The support of L yields a finite subset of V and is defined by:

(Def. 5) The support of $L = \{v : L(v) \neq 0_R\}$.

Next we state two propositions:

(19)[2] $x \in$ the support of L iff there exists v such that $x = v$ and $L(v) \neq 0_R$.

(20) $L(v) = 0_R$ iff $v \notin$ the support of L.

Let us consider R and let us consider V. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yields a linear combination of V and is defined by:

(Def. 6) The support of $\mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

The following proposition is true

(22)[3] $\mathbf{0}_{\mathrm{LC}_V}(v) = 0_R$.

Let us consider R, let us consider V, and let us consider A. A linear combination of V is said to be a linear combination of A if:

(Def. 7) The support of it $\subseteq A$.

In the sequel l is a linear combination of A.

The following four propositions are true:

(25)[4] If $A \subseteq B$, then l is a linear combination of B.

(26) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(27) For every linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(28) L is a linear combination of the support of L.

Let us consider R, let us consider V, let us consider F, and let us consider f. The functor $f\,F$ yielding a finite sequence of elements of the carrier of V is defined as follows:

(Def. 8) $\mathrm{len}(f\,F) = \mathrm{len}\,F$ and for every i such that $i \in \mathrm{dom}(f\,F)$ holds $(f\,F)(i) = F_i \cdot f(F_i)$.

We now state several propositions:

(32)[5] If $i \in \mathrm{dom}\,F$ and $v = F(i)$, then $(f\,F)(i) = v \cdot f(v)$.

(33) $f\,\varepsilon_{(\text{the carrier of } V)} = \varepsilon_{(\text{the carrier of } V)}$.

[2] The propositions (16)–(18) have been removed.
[3] The proposition (21) has been removed.
[4] The propositions (23) and (24) have been removed.
[5] The propositions (29)–(31) have been removed.

(34) $f\,\langle v\rangle = \langle v \cdot f(v)\rangle$.

(35) $f\,\langle v_1, v_2\rangle = \langle v_1 \cdot f(v_1), v_2 \cdot f(v_2)\rangle$.

(36) $f\,\langle v_1, v_2, v_3\rangle = \langle v_1 \cdot f(v_1), v_2 \cdot f(v_2), v_3 \cdot f(v_3)\rangle$.

(37) $f\,(F \frown G) = (f\,F) \frown (f\,G)$.

Let us consider R, let us consider V, and let us consider L. The functor $\sum L$ yielding a vector of V is defined as follows:

(Def. 9) There exists F such that F is one-to-one and $\mathrm{rng}\,F =$ the support of L and $\sum L = \sum(L\,F)$.

The following propositions are true:

(40)[6] If $0_R \neq \mathbf{1}_R$, then $A \neq \emptyset$ and A is linearly closed iff for every l holds $\sum l \in A$.

(41) $\sum(\mathbf{0}_{\mathrm{LC}_V}) = 0_V$.

(42) For every linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $\sum l = 0_V$.

(43) For every linear combination l of $\{v\}$ holds $\sum l = v \cdot l(v)$.

(44) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = v_1 \cdot l(v_1) + v_2 \cdot l(v_2)$.

(45) If the support of $L = \emptyset$, then $\sum L = 0_V$.

(46) If the support of $L = \{v\}$, then $\sum L = v \cdot L(v)$.

(47) If the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = v_1 \cdot L(v_1) + v_2 \cdot L(v_2)$.

Let us consider R, let us consider V, and let us consider L_1, L_2. Let us observe that $L_1 = L_2$ if and only if:

(Def. 10) For every v holds $L_1(v) = L_2(v)$.

Let us consider R, let us consider V, and let us consider L_1, L_2. The functor $L_1 + L_2$ yields a linear combination of V and is defined by:

(Def. 11) For every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

Next we state several propositions:

(51)[7] The support of $L_1 + L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(52) Suppose L_1 is a linear combination of A and L_2 is a linear combination of A. Then $L_1 + L_2$ is a linear combination of A.

(53) Let R be a commutative ring, V be a right module over R, and L_1, L_2 be linear combinations of V. Then $L_1 + L_2 = L_2 + L_1$.

(54) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(55) Let R be a commutative ring, V be a right module over R, and L be a linear combination of V. Then $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider R, let us consider V, a, and let us consider L. The functor $L \cdot a$ yielding a linear combination of V is defined by:

(Def. 12) For every v holds $(L \cdot a)(v) = L(v) \cdot a$.

Next we state the proposition

[6] The propositions (38) and (39) have been removed.
[7] The propositions (48)–(50) have been removed.

(58)[8] The support of $L \cdot a \subseteq$ the support of L.

In the sequel R_1 is an integral domain, V_1 is a right module over R_1, L_4 is a linear combination of V_1, and a_1 is a scalar of R_1.

The following propositions are true:

(59) If $a_1 \neq 0_{(R_1)}$, then the support of $L_4 \cdot a_1 =$ the support of L_4.

(60) $L \cdot 0_R = \mathbf{0}_{\mathrm{LC}_V}$.

(61) If L is a linear combination of A, then $L \cdot a$ is a linear combination of A.

(62) $L \cdot (a+b) = L \cdot a + L \cdot b$.

(63) $(L_1 + L_2) \cdot a = L_1 \cdot a + L_2 \cdot a$.

(64) $(L \cdot b) \cdot a = L \cdot (b \cdot a)$.

(65) $L \cdot \mathbf{1}_R = L$.

Let us consider R, let us consider V, and let us consider L. The functor $-L$ yielding a linear combination of V is defined as follows:

(Def. 13) $-L = L \cdot -\mathbf{1}_R$.

Let us note that the functor $-L$ is involutive.

One can prove the following propositions:

(67)[9] $(-L)(v) = -L(v)$.

(68) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(69) The support of $-L =$ the support of L.

(70) If L is a linear combination of A, then $-L$ is a linear combination of A.

Let us consider R, let us consider V, and let us consider L_1, L_2. The functor $L_1 - L_2$ yielding a linear combination of V is defined by:

(Def. 14) $L_1 - L_2 = L_1 + -L_2$.

The following propositions are true:

(73)[10] $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(74) The support of $L_1 - L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(75) Suppose L_1 is a linear combination of A and L_2 is a linear combination of A. Then $L_1 - L_2$ is a linear combination of A.

(76) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

(77) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

For simplicity, we adopt the following rules: R denotes an integral domain, V denotes a right module over R, L, L_1, L_2 denote linear combinations of V, and a denotes a scalar of R.

The following propositions are true:

(78) $\sum(L \cdot a) = \sum L \cdot a$.

(79) $\sum(-L) = -\sum L$.

(80) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

[8] The propositions (56) and (57) have been removed.
[9] The proposition (66) has been removed.
[10] The propositions (71) and (72) have been removed.

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