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Linear Combinations in Real Linear Space

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Summary. The article is continuation of [17]. At the beginning we prove some theorems concerning sums of finite sequence of vectors. We introduce the following notions: sum of finite subset of vectors, linear combination, carrier of linear combination, linear combination of elements of a given set of vectors, sum of linear combination. We also show that the set of linear combinations is a real linear space. At the end of article we prove some auxiliary theorems that should be proved in [8], [5], [9], [2] or [10].

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The articles [13], [12], [7], [19], [15], [9], [3], [20], [5], [6], [17], [10], [16], [14], [4], [18], [1], and [11] provide the notation and terminology for this paper.

In this article we present several logical schemes. The scheme *LambdaSep1* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

> There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{D}$ and for every element x of $\mathcal{A}$ such that $x \neq \mathcal{C}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

The scheme *LambdaSep2* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, elements $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, elements $\mathcal{E}$, $\mathcal{F}$ of $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

> There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that $f(\mathcal{C}) = \mathcal{E}$ and $f(\mathcal{D}) = \mathcal{F}$ and for every element x of $\mathcal{A}$ such that $x \neq \mathcal{C}$ and $x \neq \mathcal{D}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters satisfy the following condition:

- $\mathcal{C} \neq \mathcal{D}$.

For simplicity, we adopt the following rules: X, Y, x are sets, i, k, n are natural numbers, V is a real linear space, v, v_1, v_2, v_3 are vectors of V, a, b are real numbers, F, G are finite sequences of elements of the carrier of V, A, B are subsets of V, and f is a function from the carrier of V into $\mathbb{R}$.

Let S be a 1-sorted structure and let us consider x. Let us assume that $x \in S$. The functor x^S yields an element of S and is defined as follows:

(Def. 1) $x^S = x$.

The following propositions are true:

(3)[1] For every non empty 1-sorted structure S and for every element v of S holds $v^S = v$.

(4) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and F, G, H be finite sequences of elements of the carrier of V. Suppose $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{dom} F$ holds $H(k) = F_k + G_k$. Then $\sum H = \sum F + \sum G$.

[1] The propositions (1) and (2) have been removed.

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(5) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{dom} F$ holds $G(k) = a \cdot F_k$, then $\sum G = a \cdot \sum F$.

(6) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and F, G be finite sequences of elements of the carrier of V. If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{dom} F$ holds $G(k) = -F_k$, then $\sum G = -\sum F$.

(7) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and F, G, H be finite sequences of elements of the carrier of V. Suppose $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{dom} F$ holds $H(k) = F_k - G_k$. Then $\sum H = \sum F - \sum G$.

(8) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, F, G be finite sequences of elements of the carrier of V, and f be a permutation of $\operatorname{dom} F$. If $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = F(f(i))$, then $\sum F = \sum G$.

(9) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, F, G be finite sequences of elements of the carrier of V, and f be a permutation of $\operatorname{dom} F$. If $G = F \cdot f$, then $\sum F = \sum G$.

Let V be a 1-sorted structure. One can check that there exists a subset of V which is empty and finite.

Let V be a 1-sorted structure and let S, T be finite subsets of V. Then $S \cup T$ is a finite subset of V. Then $S \cap T$ is a finite subset of V. Then $S \setminus T$ is a finite subset of V. Then $S \dot{-} T$ is a finite subset of V.

Let V be a non empty loop structure and let T be a finite subset of V. Let us assume that V is Abelian, add-associative, and right zeroed. The functor $\sum T$ yields an element of V and is defined by:

(Def. 4)[2] There exists a finite sequence F of elements of the carrier of V such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

Let V be a non empty 1-sorted structure. Observe that there exists a subset of V which is non empty and finite.

Let V be a non empty 1-sorted structure and let v be an element of V. Then $\{v\}$ is a finite subset of V.

Let V be a non empty 1-sorted structure and let v_1, v_2 be elements of V. Then $\{v_1, v_2\}$ is a finite subset of V.

Let V be a non empty 1-sorted structure and let v_1, v_2, v_3 be elements of V. Then $\{v_1, v_2, v_3\}$ is a finite subset of V.

The following propositions are true:

(14)[3] For every Abelian add-associative right zeroed non empty loop structure V holds $\sum(\emptyset_V) = 0_V$.

(15) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $\sum\{v\} = v$.

(16) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v_1, v_2 be elements of V. If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(17) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v_1, v_2, v_3 be elements of V. If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = v_1 + v_2 + v_3$.

(18) Let V be an Abelian add-associative right zeroed non empty loop structure and S, T be finite subsets of V. If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

[2] The definitions (Def. 2) and (Def. 3) have been removed.

[3] The propositions (10)–(13) have been removed.

(19) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and S, T be finite subsets of V. Then $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(20) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and S, T be finite subsets of V. Then $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

(21) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and S, T be finite subsets of V. Then $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(22) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and S, T be finite subsets of V. Then $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(23) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and S, T be finite subsets of V. Then $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(24) Let V be an Abelian add-associative right zeroed non empty loop structure and S, T be finite subsets of V. Then $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let V be a non empty zero structure. An element of $\mathbb{R}^{\text{the carrier of } V}$ is said to be a linear combination of V if:

(Def. 5) There exists a finite subset T of V such that for every element v of V such that $v \notin T$ holds $\text{it}(v) = 0$.

In the sequel L, L_1, L_2, L_3 denote linear combinations of V.

Let V be a non empty loop structure and let L be a linear combination of V. The support of L yielding a finite subset of V is defined as follows:

(Def. 6) The support of $L = \{v; v \text{ ranges over elements of } V\colon L(v) \neq 0\}$.

Next we state the proposition

(28)[4] Let V be a non empty loop structure, L be a linear combination of V, and v be an element of V. Then $L(v) = 0$ if and only if $v \notin$ the support of L.

Let V be a non empty loop structure. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yields a linear combination of V and is defined as follows:

(Def. 7) The support of $\mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

The following proposition is true

(30)[5] For every non empty loop structure V and for every element v of V holds $\mathbf{0}_{\mathrm{LC}_V}(v) = 0$.

Let V be a non empty loop structure and let A be a subset of V. A linear combination of V is said to be a linear combination of A if:

(Def. 8) The support of it $\subseteq A$.

In the sequel l denotes a linear combination of A.

The following three propositions are true:

(33)[6] If $A \subseteq B$, then l is a linear combination of B.

(34) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(35) For every linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

Let us consider V, let us consider F, and let us consider f. The functor $f\,F$ yields a finite sequence of elements of the carrier of V and is defined as follows:

[4] The propositions (25)–(27) have been removed.
[5] The proposition (29) has been removed.
[6] The propositions (31) and (32) have been removed.

(Def. 9) $\operatorname{len}(f\,F) = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom}(f\,F)$ holds $(f\,F)(i) = f(F_i) \cdot F_i$.

The following propositions are true:

(40)[7] If $i \in \operatorname{dom} F$ and $v = F(i)$, then $(f\,F)(i) = f(v) \cdot v$.

(41) $f\,\varepsilon_{(\text{the carrier of } V)} = \varepsilon_{(\text{the carrier of } V)}$.

(42) $f\,\langle v \rangle = \langle f(v) \cdot v \rangle$.

(43) $f\,\langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(44) $f\,\langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

Let us consider V and let us consider L. The functor $\sum L$ yields an element of V and is defined as follows:

(Def. 10) There exists F such that F is one-to-one and $\operatorname{rng} F =$ the support of L and $\sum L = \sum(L\,F)$.

Next we state several propositions:

(47)[8] $A \neq \emptyset$ and A is linearly closed iff for every l holds $\sum l \in A$.

(48) $\sum(\mathbf{0}_{\mathrm{LC}_V}) = 0_V$.

(49) For every linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $\sum l = 0_V$.

(50) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(51) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(52) If the support of $L = \emptyset$, then $\sum L = 0_V$.

(53) If the support of $L = \{v\}$, then $\sum L = L(v) \cdot v$.

(54) If the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let V be a non empty loop structure and let L_1, L_2 be linear combinations of V. Let us observe that $L_1 = L_2$ if and only if:

(Def. 11) For every element v of V holds $L_1(v) = L_2(v)$.

Let V be a non empty loop structure and let L_1, L_2 be linear combinations of V. The functor $L_1 + L_2$ yields a linear combination of V and is defined as follows:

(Def. 12) For every element v of V holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

The following propositions are true:

(58)[9] The support of $L_1 + L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(59) Suppose L_1 is a linear combination of A and L_2 is a linear combination of A. Then $L_1 + L_2$ is a linear combination of A.

(60) For every non empty loop structure V and for all linear combinations L_1, L_2 of V holds $L_1 + L_2 = L_2 + L_1$.

(61) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(62) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

[7] The propositions (36)–(39) have been removed.
[8] The propositions (45) and (46) have been removed.
[9] The propositions (55)–(57) have been removed.

Let us consider V, a and let us consider L. The functor $a \cdot L$ yielding a linear combination of V is defined as follows:

(Def. 13) For every v holds $(a \cdot L)(v) = a \cdot L(v)$.

The following propositions are true:

(65)[10] If $a \neq 0$, then the support of $a \cdot L =$ the support of L.

(66) $0 \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(67) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(68) $(a+b) \cdot L = a \cdot L + b \cdot L$.

(69) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(70) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(71) $1 \cdot L = L$.

Let us consider V, L. The functor $-L$ yields a linear combination of V and is defined by:

(Def. 14) $-L = (-1) \cdot L$.

One can prove the following propositions:

(73)[11] $(-L)(v) = -L(v)$.

(74) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(75) The support of $-L =$ the support of L.

(76) If L is a linear combination of A, then $-L$ is a linear combination of A.

(77) $--L = L$.

Let us consider V and let us consider L_1, L_2. The functor $L_1 - L_2$ yields a linear combination of V and is defined as follows:

(Def. 15) $L_1 - L_2 = L_1 + -L_2$.

The following four propositions are true:

(79)[12] $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(80) The support of $L_1 - L_2 \subseteq$ (the support of L_1) $\cup$ (the support of L_2).

(81) Suppose L_1 is a linear combination of A and L_2 is a linear combination of A. Then $L_1 - L_2$ is a linear combination of A.

(82) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

Let us consider V. The functor LC_V yielding a set is defined as follows:

(Def. 16) $x \in \mathrm{LC}_V$ iff x is a linear combination of V.

Let us consider V. Note that LC_V is non empty.

In the sequel e, e_1, e_2 denote elements of LC_V.

Let us consider V and let us consider e. The functor ${}^{@}e$ yielding a linear combination of V is defined as follows:

[10] The propositions (63) and (64) have been removed.
[11] The proposition (72) has been removed.
[12] The proposition (78) has been removed.

(Def. 17) $^{@}e = e$.

Let us consider V and let us consider L. The functor $^{@}L$ yields an element of LC_V and is defined as follows:

(Def. 18) $^{@}L = L$.

Let us consider V. The functor $+_{\mathrm{LC}_V}$ yields a binary operation on LC_V and is defined by:

(Def. 19) For all e_1, e_2 holds $+_{\mathrm{LC}_V}(e_1, e_2) = (^{@}e_1) + {}^{@}e_2$.

Let us consider V. The functor $\cdot_{\mathrm{LC}_V}$ yielding a function from $[:\mathbb{R}, \mathrm{LC}_V:]$ into LC_V is defined as follows:

(Def. 20) For all a, e holds $\cdot_{\mathrm{LC}_V}(\langle a, e\rangle) = a \cdot (^{@}e)$.

Let us consider V. The functor $\mathbb{LC}_V$ yields a real linear space and is defined as follows:

(Def. 21) $\mathbb{LC}_V = \langle \mathrm{LC}_V, {}^{@}(\mathbf{0}_{\mathrm{LC}_V}), +_{\mathrm{LC}_V}, \cdot_{\mathrm{LC}_V}\rangle$.

Let us consider V. One can verify that $\mathbb{LC}_V$ is strict and non empty.
The following propositions are true:

(92)[13] The carrier of $\mathbb{LC}_V = \mathrm{LC}_V$.

(93) The zero of $\mathbb{LC}_V = \mathbf{0}_{\mathrm{LC}_V}$.

(94) The addition of $\mathbb{LC}_V = +_{\mathrm{LC}_V}$.

(95) The external multiplication of $\mathbb{LC}_V = \cdot_{\mathrm{LC}_V}$.

(96) $L_1{}^{\mathbb{LC}_V} + L_2{}^{\mathbb{LC}_V} = L_1 + L_2$.

(97) $a \cdot L^{\mathbb{LC}_V} = a \cdot L$.

(98) $-L^{\mathbb{LC}_V} = -L$.

(99) $L_1{}^{\mathbb{LC}_V} - L_2{}^{\mathbb{LC}_V} = L_1 - L_2$.

Let us consider V and let us consider A. The functor $\mathbb{LC}_A$ yielding a strict subspace of $\mathbb{LC}_V$ is defined by:

(Def. 22) The carrier of $\mathbb{LC}_A = \{l\}$.

Next we state three propositions:

(103)[14] If $k < n$, then $n - 1$ is a natural number.

(107)[15] If X is finite and Y is finite, then $X \dot{-} Y$ is finite.

(108) For every function f such that $f^{-1}(X) = f^{-1}(Y)$ and $X \subseteq \mathrm{rng}\, f$ and $Y \subseteq \mathrm{rng}\, f$ holds $X = Y$.

[13] The propositions (83)–(91) have been removed.
[14] The propositions (100)–(102) have been removed.
[15] The propositions (104)–(106) have been removed.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Library Committee. Boolean properties of sets — requirements. *Journal of Formalized Mathematics*, EMM, 2002. `http://mizar.org/JFM/EMM/boole.html`.

[9] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[12] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/enumset1.html`.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[14] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fraenkel.html`.

[15] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[16] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlsub_1.html`.

[17] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[18] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[19] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

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