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Vectors in Real Linear Space

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Summary. In this article we introduce a notion of real linear space, operations on vectors: addition, multiplication by real number, inverse vector, subtraction. The sum of finite sequence of the vectors is also defined. Theorems that belong rather to [1] or [4] are proved.

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The articles [11], [8], [13], [2], [3], [12], [10], [14], [6], [7], [5], [9], [4], and [1] provide the notation and terminology for this paper.

We introduce loop structures which are extensions of zero structure and are systems

⟨ a carrier, an addition, a zero ⟩,

where the carrier is a set, the addition is a binary operation on the carrier, and the zero is an element of the carrier.

We consider RLS structures as extensions of loop structure as systems

⟨ a carrier, a zero, an addition, an external multiplication ⟩,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from $[:\mathbb{R}, \text{the carrier}:]$ into the carrier.

Let us note that there exists an RLS structure which is non empty.

In the sequel V is a non empty RLS structure and x is a set.

Let V be an RLS structure. A vector of V is an element of V.

Let V be a 1-sorted structure and let us consider x. The predicate $x \in V$ is defined by:

(Def. 1) $x \in$ the carrier of V.

The following proposition is true

(3)[1] For every non empty 1-sorted structure V and for every element v of V holds $v \in V$.

Let V be a zero structure. The functor 0_V yielding an element of V is defined by:

(Def. 2) $0_V =$ the zero of V.

In the sequel v denotes a vector of V and a, b denote real numbers.

Let us observe that there exists a loop structure which is strict and non empty.

Let V be a non empty loop structure and let v, w be elements of V. The functor $v+w$ yields an element of V and is defined as follows:

(Def. 3) $v+w = (\text{the addition of } V)(\langle v, w\rangle)$.

Let us consider V, let us consider v, and let us consider a. The functor $a \cdot v$ yields an element of V and is defined as follows:

[1] The propositions (1) and (2) have been removed.

(Def. 4) $a \cdot v =$ (the external multiplication of V)($\langle a, v \rangle$).

We now state the proposition

(5)[2] For every non empty loop structure V and for all elements v, w of V holds $v + w =$ (the addition of V)(v, w).

Let Z_1 be a non empty set, let O be an element of Z_1, let F be a binary operation on Z_1, and let G be a function from $[:\mathbb{R}, Z_1:]$ into Z_1. Observe that $\langle Z_1, O, F, G \rangle$ is non empty.

Let I_1 be a non empty loop structure. We say that I_1 is Abelian if and only if:

(Def. 5) For all elements v, w of I_1 holds $v + w = w + v$.

We say that I_1 is add-associative if and only if:

(Def. 6) For all elements u, v, w of I_1 holds $(u + v) + w = u + (v + w)$.

We say that I_1 is right zeroed if and only if:

(Def. 7) For every element v of I_1 holds $v + 0_{(I_1)} = v$.

We say that I_1 is right complementable if and only if:

(Def. 8) For every element v of I_1 there exists an element w of I_1 such that $v + w = 0_{(I_1)}$.

Let I_1 be a non empty RLS structure. We say that I_1 is real linear space-like if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) For every a and for all vectors v, w of I_1 holds $a \cdot (v + w) = a \cdot v + a \cdot w$,

(ii) for all a, b and for every vector v of I_1 holds $(a + b) \cdot v = a \cdot v + b \cdot v$,

(iii) for all a, b and for every vector v of I_1 holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$, and

(iv) for every vector v of I_1 holds $1 \cdot v = v$.

Let us note that there exists a non empty loop structure which is strict, Abelian, add-associative, right zeroed, and right complementable.

Let us observe that there exists a non empty RLS structure which is non empty, strict, Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

A real linear space is an Abelian add-associative right zeroed right complementable real linear space-like non empty RLS structure.

Let V be an Abelian non empty loop structure and let v, w be elements of V. Let us note that the functor $v + w$ is commutative.

The following proposition is true

(7)[3] Suppose that for all vectors v, w of V holds $v + w = w + v$ and for all vectors u, v, w of V holds $(u + v) + w = u + (v + w)$ and for every vector v of V holds $v + 0_V = v$ and for every vector v of V there exists a vector w of V such that $v + w = 0_V$ and for every a and for all vectors v, w of V holds $a \cdot (v + w) = a \cdot v + a \cdot w$ and for all a, b and for every vector v of V holds $(a + b) \cdot v = a \cdot v + b \cdot v$ and for all a, b and for every vector v of V holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$ and for every vector v of V holds $1 \cdot v = v$. Then V is a real linear space.

In the sequel V is a real linear space and v, w are vectors of V.

The following proposition is true

(10)[4] Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v + 0_V = v$ and $0_V + v = v$.

[2] The proposition (4) has been removed.
[3] The proposition (6) has been removed.
[4] The propositions (8) and (9) have been removed.

Let V be a non empty loop structure and let v be an element of V. Let us assume that V is add-associative, right zeroed, and right complementable. The functor $-v$ yields an element of V and is defined as follows:

(Def. 10) $v + -v = 0_V$.

Let V be a non empty loop structure and let v, w be elements of V. The functor $v - w$ yielding an element of V is defined as follows:

(Def. 11) $v - w = v + -w$.

We now state a number of propositions:

(16)[5] Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v + -v = 0_V$ and $-v + v = 0_V$.

(19)[6] Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $v + w = 0_V$, then $v = -w$.

(20) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u be elements of V. Then there exists an element w of V such that $v + w = u$.

(21) Let V be an add-associative right zeroed right complementable non empty loop structure and w, u, v_1, v_2 be elements of V. If $w + v_1 = w + v_2$ or $v_1 + w = v_2 + w$, then $v_1 = v_2$.

(22) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $v + w = v$ or $w + v = v$, then $w = 0_V$.

(23) If $a = 0$ or $v = 0_V$, then $a \cdot v = 0_V$.

(24) If $a \cdot v = 0_V$, then $a = 0$ or $v = 0_V$.

(25) For every add-associative right zeroed right complementable non empty loop structure V holds $-0_V = 0_V$.

(26) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v - 0_V = v$.

(27) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $0_V - v = -v$.

(28) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $v - v = 0_V$.

(29) $-v = (-1) \cdot v$.

(30) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $--v = v$.

(31) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $-v = -w$, then $v = w$.

(33)[7] If $v = -v$, then $v = 0_V$.

(34) If $v + v = 0_V$, then $v = 0_V$.

(35) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. If $v - w = 0_V$, then $v = w$.

(36) Let V be an add-associative right zeroed right complementable non empty loop structure and u, v be elements of V. Then there exists an element w of V such that $v - w = u$.

[5] The propositions (11)–(15) have been removed.
[6] The propositions (17) and (18) have been removed.
[7] The proposition (32) has been removed.

(37) Let V be an add-associative right zeroed right complementable non empty loop structure and w, v_1, v_2 be elements of V. If $w - v_1 = w - v_2$, then $v_1 = v_2$.

(38) $a \cdot -v = (-a) \cdot v$.

(39) $a \cdot -v = -a \cdot v$.

(40) $(-a) \cdot -v = a \cdot v$.

(41) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $v - (u + w) = v - w - u$.

(42) For every add-associative non empty loop structure V and for all elements v, u, w of V holds $(v + u) - w = v + (u - w)$.

(43) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $v - (u - w) = (v - u) + w$.

(44) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-(v + w) = -w - v$.

(45) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-(v + w) = -w + -v$.

(46) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-v - w = -w - v$.

(47) Let V be an add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $-(v - w) = w + -v$.

(48) $a \cdot (v - w) = a \cdot v - a \cdot w$.

(49) $(a - b) \cdot v = a \cdot v - b \cdot v$.

(50) If $a \neq 0$ and $a \cdot v = a \cdot w$, then $v = w$.

(51) If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

Let V be a non empty 1-sorted structure and let v, u be elements of V. Then $\langle v, u \rangle$ is a finite sequence of elements of the carrier of V.

Let V be a non empty 1-sorted structure and let v, u, w be elements of V. Then $\langle v, u, w \rangle$ is a finite sequence of elements of the carrier of V.

For simplicity, we adopt the following rules: V is a non empty loop structure, F, G are finite sequences of elements of the carrier of V, f is a function from $\mathbb{N}$ into the carrier of V, v is an element of V, and j, k, n are natural numbers.

Let us consider V and let us consider F. The functor $\sum F$ yields an element of V and is defined by:

(Def. 12) There exists f such that $\sum F = f(\mathrm{len}\, F)$ and $f(0) = 0_V$ and for all j, v such that $j < \mathrm{len}\, F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

The following two propositions are true:

(54)[8] If $k \in \mathrm{Seg}\, n$ and $\mathrm{len}\, F = n$, then $F(k)$ is an element of V.

(55) If $\mathrm{len}\, F = \mathrm{len}\, G + 1$ and $G = F \restriction \mathrm{dom}\, G$ and $v = F(\mathrm{len}\, F)$, then $\sum F = \sum G + v$.

In the sequel V is a real linear space, v is a vector of V, and F, G are finite sequences of elements of the carrier of V.

The following three propositions are true:

[8] The propositions (52) and (53) have been removed.

(56) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = a \cdot v$, then $\sum F = a \cdot \sum G$.

(57) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and F, G be finite sequences of elements of the carrier of V. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every k and for every element v of V such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = -v$. Then $\sum F = -\sum G$.

(58) Let V be an add-associative right zeroed non empty loop structure and F, G be finite sequences of elements of the carrier of V. Then $\sum(F \frown G) = \sum F + \sum G$.

For simplicity, we use the following convention: V denotes an add-associative right zeroed right complementable non empty loop structure, F denotes a finite sequence of elements of the carrier of V, v, v_1, v_2, u, w denote elements of V, and p, q denote finite sequences.

We now state a number of propositions:

(59) Let V be an Abelian add-associative right zeroed non empty loop structure and F, G be finite sequences of elements of the carrier of V. If $\operatorname{rng} F = \operatorname{rng} G$ and F is one-to-one and G is one-to-one, then $\sum F = \sum G$.

(60) For every non empty loop structure V holds $\sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(61) Let V be an add-associative right zeroed right complementable non empty loop structure and v be an element of V. Then $\sum\langle v\rangle = v$.

(62) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u be elements of V. Then $\sum\langle v, u\rangle = v + u$.

(63) Let V be an add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle v, u, w\rangle = v + u + w$.

(64) For every real linear space V and for every real number a holds $a \cdot \sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(66)[9] For every real linear space V and for every real number a and for all vectors v, u of V holds $a \cdot \sum\langle v, u\rangle = a \cdot v + a \cdot u$.

(67) Let V be a real linear space, a be a real number, and v, u, w be vectors of V. Then $a \cdot \sum\langle v, u, w\rangle = a \cdot v + a \cdot u + a \cdot w$.

(68) $-\sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(69) $-\sum\langle v\rangle = -v$.

(70) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u be elements of V. Then $-\sum\langle v, u\rangle = -v - u$.

(71) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $-\sum\langle v, u, w\rangle = -v - u - w$.

(72) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, w be elements of V. Then $\sum\langle v, w\rangle = \sum\langle w, v\rangle$.

(73) $\sum\langle v, w\rangle = \sum\langle v\rangle + \sum\langle w\rangle$.

(74) $\sum\langle 0_V, 0_V\rangle = 0_V$.

(75) $\sum\langle 0_V, v\rangle = v$ and $\sum\langle v, 0_V\rangle = v$.

(76) $\sum\langle v, -v\rangle = 0_V$ and $\sum\langle -v, v\rangle = 0_V$.

(77) $\sum\langle v, -w\rangle = v - w$.

[9] The proposition (65) has been removed.

(78) $\sum\langle -v, -w\rangle = -(w+v)$.

(79) For every real linear space V and for every vector v of V holds $\sum\langle v, v\rangle = 2\cdot v$.

(80) For every real linear space V and for every vector v of V holds $\sum\langle -v, -v\rangle = (-2)\cdot v$.

(81) $\sum\langle u, v, w\rangle = \sum\langle u\rangle + \sum\langle v\rangle + \sum\langle w\rangle$.

(82) $\sum\langle u, v, w\rangle = \sum\langle u, v\rangle + w$.

(83) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle v, w\rangle + u$.

(84) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle u, w\rangle + v$.

(85) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle u, w, v\rangle$.

(86) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle v, u, w\rangle$.

(87) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle v, w, u\rangle$.

(89)[10] Let V be an Abelian add-associative right zeroed right complementable non empty loop structure and v, u, w be elements of V. Then $\sum\langle u, v, w\rangle = \sum\langle w, v, u\rangle$.

(90) $\sum\langle 0_V, 0_V, 0_V\rangle = 0_V$.

(91) $\sum\langle 0_V, 0_V, v\rangle = v$ and $\sum\langle 0_V, v, 0_V\rangle = v$ and $\sum\langle v, 0_V, 0_V\rangle = v$.

(92) $\sum\langle 0_V, u, v\rangle = u+v$ and $\sum\langle u, v, 0_V\rangle = u+v$ and $\sum\langle u, 0_V, v\rangle = u+v$.

(93) For every real linear space V and for every vector v of V holds $\sum\langle v, v, v\rangle = 3\cdot v$.

(94) If $\operatorname{len} F = 0$, then $\sum F = 0_V$.

(95) If $\operatorname{len} F = 1$, then $\sum F = F(1)$.

(96) If $\operatorname{len} F = 2$ and $v_1 = F(1)$ and $v_2 = F(2)$, then $\sum F = v_1 + v_2$.

(97) If $\operatorname{len} F = 3$ and $v_1 = F(1)$ and $v_2 = F(2)$ and $v = F(3)$, then $\sum F = v_1 + v_2 + v$.

Let R be a non empty zero structure and let a be an element of R. We say that a is non-zero if and only if:

(Def. 13) $a \neq 0_R$.

In the sequel j, k, n denote natural numbers.
One can prove the following propositions:

(98) If $j < 1$, then $j = 0$.

(99) $1 \leq k$ iff $k \neq 0$.

(102)[11] If $k \neq 0$, then $n < n+k$.

(103) $k < k+n$ iff $1 \leq n$.

(104) $\operatorname{Seg} k = \operatorname{Seg}(k+1) \setminus \{k+1\}$.

(105) $p = (p \frown q) \restriction \operatorname{dom} p$.

(106) If $\operatorname{rng} p = \operatorname{rng} q$ and p is one-to-one and q is one-to-one, then $\operatorname{len} p = \operatorname{len} q$.

[10] The proposition (88) has been removed.
[11] The propositions (100) and (101) have been removed.

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