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Operations on Subspaces in Real Linear Space

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Summary. In this article the following operations on subspaces of real linear space are intoduced: sum, intersection and direct sum. Some theorems about those notions are proved. We define linear complement of a subspace. Some theorems about decomposition of a vector onto two subspaces and onto subspace and its linear complement are proved. We also show that a set of subspaces with operations sum and intersection is a lattice. At the end of the article theorems that belong rather to [4], [8], [7] or [12] are proved.

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The articles [6], [3], [9], [1], [10], [2], [12], [11], [8], [7], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: V is a real linear space, W, W_1, W_2, W_3 are subspaces of V, u, u_1, u_2, v, v_1, v_2 are vectors of V, X, Y are sets, and x is a set.

Let us consider V and let us consider W_1, W_2. The functor $W_1 + W_2$ yields a strict subspace of V and is defined as follows:

(Def. 1) The carrier of $W_1 + W_2 = \{v + u : v \in W_1 \ \wedge \ u \in W_2\}$.

Let us consider V and let us consider W_1, W_2. The functor $W_1 \cap W_2$ yields a strict subspace of V and is defined by:

(Def. 2) The carrier of $W_1 \cap W_2 =$ (the carrier of W_1) $\cap$ (the carrier of W_2).

Next we state a number of propositions:

(5)[1] $x \in W_1 + W_2$ iff there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ iff $x \in W_1$ and $x \in W_2$.

(8) For every strict subspace W of V holds $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(12) For every strict subspace W_2 of V holds W_1 is a subspace of W_2 iff $W_1 + W_2 = W_2$.

(13) For every strict subspace W of V holds $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

[1] The propositions (1)–(4) have been removed.

(14) $\mathbf{0}_V + \Omega_V =$ the RLS structure of V and $\Omega_V + \mathbf{0}_V =$ the RLS structure of V.

(15) $\Omega_V + W =$ the RLS structure of V and $W + \Omega_V =$ the RLS structure of V.

(16) For every strict real linear space V holds $\Omega_V + \Omega_V = V$.

(17) For every strict subspace W of V holds $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(21) For every strict subspace W_1 of V holds W_1 is a subspace of W_2 iff $W_1 \cap W_2 = W_1$.

(22) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(23) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(24) For every strict subspace W of V holds $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(25) For every strict real linear space V holds $\Omega_V \cap \Omega_V = V$.

(26) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(27) For every strict subspace W_2 of V holds $W_1 \cap W_2 + W_2 = W_2$.

(28) For every strict subspace W_1 of V holds $W_1 \cap (W_1 + W_2) = W_1$.

(29) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(30) If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(31) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(32) If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(33) If W_1 is a strict subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(34) For all strict subspaces W_1, W_2 of V holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(35) For all strict subspaces W_2, W_3 of V such that W_1 is a subspace of W_2 holds $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(36) There exists W such that the carrier of $W =$ (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

Let us consider V. The functor Subspaces V yields a set and is defined as follows:

(Def. 3) For every x holds $x \in$ Subspaces V iff x is a strict subspace of V.

Let us consider V. Observe that Subspaces V is non empty.
One can prove the following proposition

(39)[2] For every strict real linear space V holds $V \in$ Subspaces V.

Let us consider V and let us consider W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def. 4) The RLS structure of $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let V be a real linear space and let W be a subspace of V. A subspace of V is called a linear complement of W if:

[2] The propositions (37) and (38) have been removed.

(Def. 5) V is the direct sum of it and W.

Let V be a real linear space and let W be a subspace of V. Observe that there exists a linear complement of W which is strict.

We now state several propositions:

(42)[3] Let V be a real linear space and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Then W_2 is a linear complement of W_1.

(43) Let V be a real linear space, W be a subspace of V, and L be a linear complement of W. Then V is the direct sum of L and W and the direct sum of W and L.

(44) Let V be a real linear space, W be a subspace of V, and L be a linear complement of W. Then $W+L=$ the RLS structure of V and $L+W=$ the RLS structure of V.

(45) Let V be a real linear space, W be a subspace of V, and L be a linear complement of W. Then $W\cap L=\mathbf{0}_V$ and $L\cap W=\mathbf{0}_V$.

(46) If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(47) Every real linear space V is the direct sum of $\mathbf{0}_V$ and Ω_V and the direct sum of Ω_V and $\mathbf{0}_V$.

(48) Let V be a real linear space, W be a subspace of V, and L be a linear complement of W. Then W is a linear complement of L.

(49) For every real linear space V holds $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

In the sequel C is a coset of W, C_1 is a coset of W_1, and C_2 is a coset of W_2.

The following propositions are true:

(50) If C_1 meets C_2, then $C_1\cap C_2$ is a coset of $W_1\cap W_2$.

(51) Let V be a real linear space and W_1, W_2 be subspaces of V. Then V is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists a vector v of V such that $C_1\cap C_2=\{v\}$.

(52) Let V be a real linear space and W_1, W_2 be subspaces of V. Then $W_1+W_2=$ the RLS structure of V if and only if for every vector v of V there exist vectors v_1, v_2 of V such that $v_1\in W_1$ and $v_2\in W_2$ and $v=v_1+v_2$.

(53) If V is the direct sum of W_1 and W_2 and $v=v_1+v_2$ and $v=u_1+u_2$ and $v_1\in W_1$ and $u_1\in W_1$ and $v_2\in W_2$ and $u_2\in W_2$, then $v_1=u_1$ and $v_2=u_2$.

(54) Suppose $V=W_1+W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v=v_1+v_2$ and $v=u_1+u_2$ and $v_1\in W_1$ and $u_1\in W_1$ and $v_2\in W_2$ and $u_2\in W_2$ holds $v_1=u_1$ and $v_2=u_2$. Then V is the direct sum of W_1 and W_2.

Let us consider V, let us consider v, and let us consider W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v_{\langle W_1,W_2\rangle}$ yielding an element of $[\!:$the carrier of V, the carrier of $V\,:\!]$ is defined as follows:

(Def. 6) $v=(v_{\langle W_1,W_2\rangle})_{\mathbf{1}}+(v_{\langle W_1,W_2\rangle})_{\mathbf{2}}$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}}\in W_1$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}}\in W_2$.

The following propositions are true:

(59)[4] If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}}=(v_{\langle W_2,W_1\rangle})_{\mathbf{2}}$.

(60) If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}}=(v_{\langle W_2,W_1\rangle})_{\mathbf{1}}$.

[3] The propositions (40) and (41) have been removed.
[4] The propositions (55)–(58) have been removed.

(61) Let V be a real linear space, W be a subspace of V, L be a linear complement of W, v be a vector of V, and t be an element of $[:$the carrier of V, the carrier of $V:]$. If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v_{\langle W,L \rangle}$.

(62) Let V be a real linear space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} + (v_{\langle W,L \rangle})_{\mathbf{2}} = v$.

(63) Let V be a real linear space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} \in W$ and $(v_{\langle W,L \rangle})_{\mathbf{2}} \in L$.

(64) Let V be a real linear space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} = (v_{\langle L,W \rangle})_{\mathbf{2}}$.

(65) Let V be a real linear space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{2}} = (v_{\langle L,W \rangle})_{\mathbf{1}}$.

In the sequel A_1, A_2 denote elements of Subspaces V.

Let us consider V. The functor SubJoin V yielding a binary operation on Subspaces V is defined as follows:

(Def. 7) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\,V)(A_1, A_2) = W_1 + W_2$.

Let us consider V. The functor SubMeet V yields a binary operation on Subspaces V and is defined by:

(Def. 8) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\,V)(A_1, A_2) = W_1 \cap W_2$.

Let X be a non empty set and let m, u be binary operations on X. Note that $\langle X, m, u \rangle$ is non empty.

The following proposition is true

(70)[5] $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a lattice.

Let us consider V. Observe that $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lattice-like.

The following propositions are true:

(71) For every real linear space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lower-bounded.

(72) For every real linear space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is upper-bounded.

(73) For every real linear space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a bound lattice.

(74) For every real linear space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is modular.

In the sequel l is a lattice and a, b are elements of l.

One can prove the following proposition

(75) For every real linear space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is complemented.

Let us consider V. One can verify that $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lower-bounded, upper-bounded, modular, and complemented.

One can prove the following propositions:

(76) Let V be a real linear space and W_1, W_2, W_3 be strict subspaces of V. If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(77) If $X \subset Y$, then there exists x such that $x \in Y$ and $x \notin X$.

[5] The propositions (66)–(69) have been removed.

(78) Let V be an add-associative right zeroed right complementable non empty loop structure and v, v_1, v_2 be elements of V. Then $v = v_1 + v_2$ if and only if $v_1 = v - v_2$.

(79) Let V be a real linear space and W be a strict subspace of V. If for every vector v of V holds $v \in W$, then $W =$ the RLS structure of V.

(80) There exists C such that $v \in C$.

(84)[6] If for every a holds $a \sqcap b = b$, then $b = \perp_l$.

(85) If for every a holds $a \sqcup b = b$, then $b = \top_l$.

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[6] The propositions (81)–(83) have been removed.