JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Subspaces and Cosets of Subspaces in Real Linear Space

Wojciech A. Trybulec
Warsaw University

Summary. The following notions are introduced in the article: subspace of a real linear space, zero subspace and improper subspace, coset of a subspace. The relation of a subset of the vectors being linearly closed is also introduced. Basic theorems concerning those notions are proved in the article.

MML Identifier: RLSUB_1.

WWW: http://mizar.org/JFM/Vol1/rlsub_1.html

The articles [4], [3], [8], [6], [5], [1], [9], [2], and [7] provide the notation and terminology for this paper.

For simplicity, we follow the rules: V, X, Y denote real linear spaces, u, v, v_1, v_2 denote vectors of V, a denotes a real number, V_1, V_2, V_3 denote subsets of V, and x denotes a set.

Let us consider V and let us consider V_1. We say that V_1 is linearly closed if and only if:

(Def. 1) For all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

We now state several propositions:

(4)[1] If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(7) $\{0_V\}$ is linearly closed.

(8) If the carrier of $V = V_1$, then V_1 is linearly closed.

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \land u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider V. A real linear space is said to be a subspace of V if it satisfies the conditions (Def. 2).

(Def. 2)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it $=$ the zero of V,

(iii) the addition of it $=$ (the addition of V)$\restriction[\!:$the carrier of it, the carrier of it$:\!]$, and

(iv) the external multiplication of it $=$ (the external multiplication of V)$\restriction[\!:\mathbb{R}$, the carrier of it$:\!]$.

[1] The propositions (1)–(3) have been removed.

We adopt the following rules: W, W_1, W_2 denote subspaces of V and w, w_1, w_2 denote vectors of W.

One can prove the following propositions:

(16)[2] If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $0_W = 0_V$.

(20) $0_{(W_1)} = 0_{(W_2)}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $0_V \in W$.

(26) $0_{(W_1)} \in W_2$.

(27) $0_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

In the sequel D is a non empty set, d_1 is an element of D, A is a binary operation on D, and M is a function from $[:\mathbb{R}, D:]$ into D.

The following propositions are true:

(32) Suppose $V_1 = D$ and $d_1 = 0_V$ and $A = (\text{the addition of } V) \restriction [:V_1, V_1:]$ and $M = (\text{the external multiplication of } V) \restriction [:\mathbb{R}, V_1:]$. Then $\langle D, d_1, A, M \rangle$ is a subspace of V.

(33) V is a subspace of V.

(34) For all strict real linear spaces V, X such that V is a subspace of X and X is a subspace of V holds $V = X$.

(35) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(36) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a subspace of W_2.

(37) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

Let us consider V. Note that there exists a subspace of V which is strict.

One can prove the following propositions:

(38) For all strict subspaces W_1, W_2 of V such that the carrier of $W_1 =$ the carrier of W_2 holds $W_1 = W_2$.

(39) For all strict subspaces W_1, W_2 of V such that for every v holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

[2] The propositions (11)–(15) have been removed.

(40) Let V be a strict real linear space and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(41) Let V be a strict real linear space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(42) If the carrier of $W = V_1$, then V_1 is linearly closed.

(43) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

Let us consider V. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined as follows:

(Def. 3) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yielding a strict subspace of V is defined by:

(Def. 4) $\Omega_V =$ the RLS structure of V.

We now state several propositions:

(48)[3] $\mathbf{0}_W = \mathbf{0}_V$.

(49) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(50) $\mathbf{0}_W$ is a subspace of V.

(51) $\mathbf{0}_V$ is a subspace of W.

(52) $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(54)[4] Every strict real linear space V is a subspace of Ω_V.

Let us consider V and let us consider v, W. The functor $v + W$ yields a subset of V and is defined as follows:

(Def. 5) $v + W = \{v + u : u \in W\}$.

Let us consider V and let us consider W. A subset of V is called a coset of W if:

(Def. 6) There exists v such that it $= v + W$.

In the sequel B, C denote cosets of W.
Next we state a number of propositions:

(58)[5] $0_V \in v + W$ iff $v \in W$.

(59) $v \in v + W$.

(60) $0_V + W =$ the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of V.

(63) $0_V \in v + W$ iff $v + W =$ the carrier of W.

(64) $v \in W$ iff $v + W =$ the carrier of W.

(65) If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(66) If $a \neq 0$ and $a \cdot v + W =$ the carrier of W, then $v \in W$.

[3] The propositions (44)–(47) have been removed.
[4] The proposition (53) has been removed.
[5] The propositions (55)–(57) have been removed.

(67) $v \in W$ iff $-v+W=$ the carrier of W.

(68) $u \in W$ iff $v+W=v+u+W$.

(69) $u \in W$ iff $v+W=(v-u)+W$.

(70) $v \in u+W$ iff $u+W=v+W$.

(71) $v+W=-v+W$ iff $v \in W$.

(72) If $u \in v_1+W$ and $u \in v_2+W$, then $v_1+W=v_2+W$.

(73) If $u \in v+W$ and $u \in -v+W$, then $v \in W$.

(74) If $a \neq 1$ and $a \cdot v \in v+W$, then $v \in W$.

(75) If $v \in W$, then $a \cdot v \in v+W$.

(76) $-v \in v+W$ iff $v \in W$.

(77) $u+v \in v+W$ iff $u \in W$.

(78) $v-u \in v+W$ iff $u \in W$.

(79) $u \in v+W$ iff there exists v_1 such that $v_1 \in W$ and $u=v+v_1$.

(80) $u \in v+W$ iff there exists v_1 such that $v_1 \in W$ and $u=v-v_1$.

(81) There exists v such that $v_1 \in v+W$ and $v_2 \in v+W$ iff $v_1-v_2 \in W$.

(82) If $v+W=u+W$, then there exists v_1 such that $v_1 \in W$ and $v+v_1=u$.

(83) If $v+W=u+W$, then there exists v_1 such that $v_1 \in W$ and $v-v_1=u$.

(84) For all strict subspaces W_1, W_2 of V holds $v+W_1=v+W_2$ iff $W_1=W_2$.

(85) For all strict subspaces W_1, W_2 of V such that $v+W_1=u+W_2$ holds $W_1=W_2$.

(86) C is linearly closed iff $C=$ the carrier of W.

(87) For all strict subspaces W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1=C_2$ holds $W_1=W_2$.

(88) $\{v\}$ is a coset of $\mathbf{0}_V$.

(89) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1=\{v\}$.

(90) The carrier of W is a coset of W.

(91) The carrier of V is a coset of Ω_V.

(92) If V_1 is a coset of Ω_V, then $V_1=$ the carrier of V.

(93) $0_V \in C$ iff $C=$ the carrier of W.

(94) $u \in C$ iff $C=u+W$.

(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1=v$.

(96) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1=v$.

(97) There exists C such that $v_1 \in C$ and $v_2 \in C$ iff $v_1-v_2 \in W$.

(98) If $u \in B$ and $u \in C$, then $B=C$.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[5] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[6] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[7] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[9] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received July 24, 1989

Published January 2, 2004