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Category of Rings

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Summary. We define the category of non-associative rings. The carriers of the rings are included in a universum. The universum is a parameter of the category.

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The articles [9], [4], [12], [13], [1], [10], [2], [11], [5], [6], [3], [8], and [7] provide the notation and terminology for this paper.

In this paper x, y are sets, D is a non empty set, and U_1 is a universal class.

Let G, H be non empty double loop structures and let I_1 be a map from G into H. We say that I_1 is linear if and only if:

(Def. 2)[1] For all scalars x, y of G holds $I_1(x+y) = I_1(x) + I_1(y)$ and for all scalars x, y of G holds $I_1(x \cdot y) = I_1(x) \cdot I_1(y)$ and $I_1(\mathbf{1}_G) = \mathbf{1}_H$.

The following proposition is true

(3)[2] Let G_1, G_2, G_3 be non empty double loop structures, f be a map from G_1 into G_2, and g be a map from G_2 into G_3. If f is linear and g is linear, then $g \cdot f$ is linear.

We consider ring morphisms structures as systems
$\langle$ a dom-map, a cod-map, a Fun $\rangle$,
where the dom-map and the cod-map are rings and the Fun is a map from the dom-map into the cod-map.

In the sequel f is a ring morphisms structure.

Let us consider f. The functor $\mathrm{dom}\, f$ yields a ring and is defined as follows:

(Def. 3) $\mathrm{dom}\, f =$ the dom-map of f.

The functor $\mathrm{cod}\, f$ yielding a ring is defined as follows:

(Def. 4) $\mathrm{cod}\, f =$ the cod-map of f.

The functor $\mathrm{fun}\, f$ yielding a map from the dom-map of f into the cod-map of f is defined as follows:

(Def. 5) $\mathrm{fun}\, f =$ the Fun of f.

In the sequel G, H, G_1, G_2, G_3, G_4 are rings.

Let G be a non empty double loop structure. One can check that id_G is linear.

Let I_1 be a ring morphisms structure. We say that I_1 is morphism of rings-like if and only if:

[1] The definition (Def. 1) has been removed.
[2] The propositions (1) and (2) have been removed.

(Def. 6) $\mathrm{fun}\, I_1$ is linear.

Let us note that there exists a ring morphisms structure which is strict and morphism of rings-like.

A morphism of rings is a morphism of rings-like ring morphisms structure.

Let us consider G. The functor I_G yields a morphism of rings and is defined by:

(Def. 7) $\mathrm{I}_G = \langle G, G, \mathrm{id}_G \rangle$.

Let us consider G. Note that I_G is strict.

In the sequel F is a morphism of rings.

Let us consider G, H. The predicate $G \leq H$ is defined by:

(Def. 8) There exists a morphism F of rings such that $\mathrm{dom}\, F = G$ and $\mathrm{cod}\, F = H$.

Let us note that the predicate $G \leq H$ is reflexive.

Let us consider G, H. Let us assume that $G \leq H$. A strict morphism of rings is said to be a morphism from G to H if:

(Def. 9) $\mathrm{dom}\, \mathrm{it} = G$ and $\mathrm{cod}\, \mathrm{it} = H$.

Let us consider G, H. Observe that there exists a morphism from G to H which is strict.

Let us consider G. Then I_G is a strict morphism from G to G.

One can prove the following three propositions:

(5)[3] Let g, f be morphisms of rings. Suppose $\mathrm{dom}\, g = \mathrm{cod}\, f$. Then there exist G_1, G_2, G_3 such that

(i) $G_1 \leq G_2$,

(ii) $G_2 \leq G_3$,

(iii) the ring morphisms structure of g is a morphism from G_2 to G_3, and

(iv) the ring morphisms structure of f is a morphism from G_1 to G_2.

(6) For every strict morphism F of rings holds F is a morphism from $\mathrm{dom}\, F$ to $\mathrm{cod}\, F$ and $\mathrm{dom}\, F \leq \mathrm{cod}\, F$.

(7) Let F be a strict morphism of rings. Then there exist G, H and there exists a map f from G into H such that F is a morphism from G to H and $F = \langle G, H, f \rangle$ and f is linear.

Let G, F be morphisms of rings. Let us assume that $\mathrm{dom}\, G = \mathrm{cod}\, F$. The functor $G \cdot F$ yielding a strict morphism of rings is defined by the condition (Def. 10).

(Def. 10) Let given G_1, G_2, G_3, g be a map from G_2 into G_3, and f be a map from G_1 into G_2. Suppose the ring morphisms structure of $G = \langle G_2, G_3, g \rangle$ and the ring morphisms structure of $F = \langle G_1, G_2, f \rangle$. Then $G \cdot F = \langle G_1, G_3, g \cdot f \rangle$.

We now state two propositions:

(8) If $G_1 \leq G_2$ and $G_2 \leq G_3$, then $G_1 \leq G_3$.

(9) Let G be a morphism from G_2 to G_3 and F be a morphism from G_1 to G_2. If $G_1 \leq G_2$ and $G_2 \leq G_3$, then $G \cdot F$ is a morphism from G_1 to G_3.

Let us consider G_1, G_2, G_3, let G be a morphism from G_2 to G_3, and let F be a morphism from G_1 to G_2. Let us assume that $G_1 \leq G_2$ and $G_2 \leq G_3$. The functor $G * F$ yielding a strict morphism from G_1 to G_3 is defined as follows:

(Def. 11) $G * F = G \cdot F$.

One can prove the following propositions:

[3] The proposition (4) has been removed.

(10) Let f, g be strict morphisms of rings. Suppose $\mathrm{dom}\, g = \mathrm{cod}\, f$. Then there exist G_1, G_2, G_3 and there exists a map f_0 from G_1 into G_2 and there exists a map g_0 from G_2 into G_3 such that $f = \langle G_1, G_2, f_0 \rangle$ and $g = \langle G_2, G_3, g_0 \rangle$ and $g \cdot f = \langle G_1, G_3, g_0 \cdot f_0 \rangle$.

(11) For all strict morphisms f, g of rings such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{dom}(g \cdot f) = \mathrm{dom}\, f$ and $\mathrm{cod}(g \cdot f) = \mathrm{cod}\, g$.

(12) Let f be a morphism from G_1 to G_2, g be a morphism from G_2 to G_3, and h be a morphism from G_3 to G_4. If $G_1 \leq G_2$ and $G_2 \leq G_3$ and $G_3 \leq G_4$, then $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(13) For all strict morphisms f, g, h of rings such that $\mathrm{dom}\, h = \mathrm{cod}\, g$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(14)(i) $\mathrm{dom}(\mathrm{I}_G) = G$,

(ii) $\mathrm{cod}(\mathrm{I}_G) = G$,

(iii) for every strict morphism f of rings such that $\mathrm{cod}\, f = G$ holds $\mathrm{I}_G \cdot f = f$, and

(iv) for every strict morphism g of rings such that $\mathrm{dom}\, g = G$ holds $g \cdot \mathrm{I}_G = g$.

Let I_1 be a set. We say that I_1 is non empty set of rings-like if and only if:

(Def. 12) Every element of I_1 is a strict ring.

Let us note that there exists a set which is non empty set of rings-like and non empty.
A non empty set of rings is a non empty set of rings-like non empty set.
In the sequel V is a non empty set of rings.
Let us consider V. We see that the element of V is a ring.
Let us consider V. Note that there exists an element of V which is strict.
Let I_1 be a set. We say that I_1 is non empty set of morphisms of rings-like if and only if:

(Def. 13) For every set x such that $x \in I_1$ holds x is a strict morphism of rings.

Let us mention that there exists a non empty set which is non empty set of morphisms of rings-like.

A non empty set of morphisms of rings is a non empty set of morphisms of rings-like non empty set.

Let M be a non empty set of morphisms of rings. We see that the element of M is a morphism of rings.

Let M be a non empty set of morphisms of rings. Note that there exists an element of M which is strict.

We now state the proposition

(17)[4] For every strict morphism f of rings holds $\{f\}$ is a non empty set of morphisms of rings.

Let us consider G, H. A non empty set of morphisms of rings is said to be a non empty set of morphisms from G into H if:

(Def. 14) Every element of it is a morphism from G to H.

Next we state two propositions:

(18) D is a non empty set of morphisms from G into H iff every element of D is a morphism from G to H.

(19) For every morphism f from G to H holds $\{f\}$ is a non empty set of morphisms from G into H.

Let us consider G, H. Let us assume that $G \leq H$. The functor $\mathrm{Morphs}(G, H)$ yielding a non empty set of morphisms from G into H is defined by:

[4] The propositions (15) and (16) have been removed.

(Def. 15) $x \in \mathrm{Morphs}(G,H)$ iff x is a morphism from G to H.

Let us consider G, H and let M be a non empty set of morphisms from G into H. We see that the element of M is a morphism from G to H.

Let us consider G, H and let M be a non empty set of morphisms from G into H. One can check that there exists an element of M which is strict.

Let us consider x, y. The predicate $\mathrm{P_{ob}}\, x, y$ is defined by the condition (Def. 16).

(Def. 16) There exist sets x_1, x_2, x_3, x_4, x_5, x_6 such that
- (i) $x = \langle\langle x_1, x_2, x_3, x_4\rangle, x_5, x_6\rangle$, and
- (ii) there exists a strict ring G such that $y = G$ and $x_1 =$ the carrier of G and $x_2 =$ the addition of G and $x_3 = \mathrm{comp}\, G$ and $x_4 =$ the zero of G and $x_5 =$ the multiplication of G and $x_6 =$ the unity of G.

One can prove the following propositions:

(20) For all sets x, y_1, y_2 such that $\mathrm{P_{ob}}\, x, y_1$ and $\mathrm{P_{ob}}\, x, y_2$ holds $y_1 = y_2$.

(21) There exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, \mathrm{Z}_3$.

Let us consider U_1. The functor $\mathrm{RingObj}(U_1)$ yields a set and is defined as follows:

(Def. 17) For every y holds $y \in \mathrm{RingObj}(U_1)$ iff there exists x such that $x \in U_1$ and $\mathrm{P_{ob}}\, x, y$.

We now state the proposition

(22) $\mathrm{Z}_3 \in \mathrm{RingObj}(U_1)$.

Let us consider U_1. Observe that $\mathrm{RingObj}(U_1)$ is non empty.

We now state the proposition

(23) Every element of $\mathrm{RingObj}(U_1)$ is a strict ring.

Let us consider U_1. One can verify that $\mathrm{RingObj}(U_1)$ is non empty set of rings-like.

Let us consider V. The functor $\mathrm{Morphs}\, V$ yields a non empty set of morphisms of rings and is defined by:

(Def. 18) $x \in \mathrm{Morphs}\, V$ iff there exist elements G, H of V such that $G \leq H$ and x is a morphism from G to H.

Let us consider V and let F be an element of $\mathrm{Morphs}\, V$. Then $\mathrm{dom}\, F$ is an element of V. Then $\mathrm{cod}\, F$ is an element of V.

Let us consider V and let G be an element of V. The functor I_G yielding a strict element of $\mathrm{Morphs}\, V$ is defined by:

(Def. 19) $\mathrm{I}_G = \mathrm{I}_G$.

Let us consider V. The functor $\mathrm{dom}\, V$ yielding a function from $\mathrm{Morphs}\, V$ into V is defined as follows:

(Def. 20) For every element f of $\mathrm{Morphs}\, V$ holds $(\mathrm{dom}\, V)(f) = \mathrm{dom}\, f$.

The functor $\mathrm{cod}\, V$ yields a function from $\mathrm{Morphs}\, V$ into V and is defined by:

(Def. 21) For every element f of $\mathrm{Morphs}\, V$ holds $(\mathrm{cod}\, V)(f) = \mathrm{cod}\, f$.

The functor I_V yielding a function from V into $\mathrm{Morphs}\, V$ is defined by:

(Def. 22) For every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

Next we state two propositions:

(24) Let g, f be elements of Morphs V. Suppose $\operatorname{dom} g = \operatorname{cod} f$. Then there exist elements G_1, G_2, G_3 of V such that $G_1 \leq G_2$ and $G_2 \leq G_3$ and g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(25) For all elements g, f of Morphs V such that $\operatorname{dom} g = \operatorname{cod} f$ holds $g \cdot f \in$ Morphs V.

Let us consider V. The functor comp V yields a partial function from [: Morphs V, Morphs V :] to Morphs V and is defined by the conditions (Def. 23).

(Def. 23)(i) For all elements g, f of Morphs V holds $\langle g, f \rangle \in \operatorname{dom} \operatorname{comp} V$ iff $\operatorname{dom} g = \operatorname{cod} f$, and

(ii) for all elements g, f of Morphs V such that $\langle g, f \rangle \in \operatorname{dom} \operatorname{comp} V$ holds $(\operatorname{comp} V)(\langle g, f \rangle) = g \cdot f$.

Let us consider U_1. The functor RingCat(U_1) yields a category structure and is defined as follows:

(Def. 24) $\operatorname{RingCat}(U_1) = \langle \operatorname{RingObj}(U_1), \operatorname{Morphs} \operatorname{RingObj}(U_1), \operatorname{dom} \operatorname{RingObj}(U_1), \operatorname{cod} \operatorname{RingObj}(U_1), \operatorname{comp} \operatorname{RingObj}(U_1), \mathrm{I}_{\operatorname{RingObj}(U_1)} \rangle$.

Let us consider U_1. Observe that RingCat(U_1) is strict.

Next we state several propositions:

(26) For all morphisms f, g of RingCat(U_1) holds $\langle g, f \rangle \in$ dom(the composition of RingCat(U_1)) iff $\operatorname{dom} g = \operatorname{cod} f$.

(27) Let f be a morphism of RingCat(U_1), f' be an element of Morphs RingObj(U_1), b be an object of RingCat(U_1), and b' be an element of RingObj(U_1). Then

(i) f is a strict element of Morphs RingObj(U_1),

(ii) f' is a morphism of RingCat(U_1),

(iii) b is a strict element of RingObj(U_1), and

(iv) b' is an object of RingCat(U_1).

(28) For every object b of RingCat(U_1) and for every element b' of RingObj(U_1) such that $b = b'$ holds $\mathrm{id}_b = \mathrm{I}_{b'}$.

(29) For every morphism f of RingCat(U_1) and for every element f' of Morphs RingObj(U_1) such that $f = f'$ holds $\operatorname{dom} f = \operatorname{dom} f'$ and $\operatorname{cod} f = \operatorname{cod} f'$.

(30) Let f, g be morphisms of RingCat(U_1) and f', g' be elements of Morphs RingObj(U_1) such that $f = f'$ and $g = g'$. Then

(i) $\operatorname{dom} g = \operatorname{cod} f$ iff $\operatorname{dom} g' = \operatorname{cod} f'$,

(ii) $\operatorname{dom} g = \operatorname{cod} f$ iff $\langle g', f' \rangle \in \operatorname{dom} \operatorname{comp} \operatorname{RingObj}(U_1)$,

(iii) if $\operatorname{dom} g = \operatorname{cod} f$, then $g \cdot f = g' \cdot f'$,

(iv) $\operatorname{dom} f = \operatorname{dom} g$ iff $\operatorname{dom} f' = \operatorname{dom} g'$, and

(v) $\operatorname{cod} f = \operatorname{cod} g$ iff $\operatorname{cod} f' = \operatorname{cod} g'$.

Let us consider U_1. Note that RingCat(U_1) is category-like.

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