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Partial Functions from a Domain to the Set of Real Numbers

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Summary. Basic operations in the set of partial functions which map a domain to the set of all real numbers are introduced. They include adition, subtraction, multiplication, division, multipication by a real number and also module. Main properties of these operations are proved. A definition of the partial function bounded on a set (bounded below and bounded above) is presented. There are theorems showing the laws of conservation of totality and boundedness for operations of partial functions. The characteristic function of a subset of a domain as a partial function is redefined and a few properties are proved.

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The articles [9], [11], [1], [10], [5], [3], [2], [8], [12], [4], [7], and [6] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: X, Y are sets, C is a non empty set, c is an element of C, f, f_1, f_2, f_3, g, g_1 are partial functions from C to $\mathbb{R}$, and r, r_1, p, p_1 are real numbers.

Next we state the proposition

(2)[1] If $0 \leq p$ and $0 \leq r$ and $p \leq p_1$ and $r \leq r_1$, then $p \cdot r \leq p_1 \cdot r_1$.

Let us consider C and let us consider f_1, f_2. The functor $\frac{f_1}{f_2}$ yields a partial function from C to $\mathbb{R}$ and is defined by:

(Def. 4)[2] $\mathrm{dom}(\frac{f_1}{f_2}) = \mathrm{dom} f_1 \cap (\mathrm{dom} f_2 \setminus {f_2}^{-1}(\{0\}))$ and for every c such that $c \in \mathrm{dom}(\frac{f_1}{f_2})$ holds $(\frac{f_1}{f_2})(c) = f_1(c) \cdot f_2(c)^{-1}$.

Let us consider C and let us consider f. The functor $\frac{1}{f}$ yielding a partial function from C to $\mathbb{R}$ is defined by:

(Def. 8)[3] $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom} f \setminus f^{-1}(\{0\})$ and for every c such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})(c) = f(c)^{-1}$.

We now state a number of propositions:

(11)[4] $\mathrm{dom}(\frac{1}{g}) \subseteq \mathrm{dom}\, g$ and $\mathrm{dom}\, g \cap (\mathrm{dom}\, g \setminus g^{-1}(\{0\})) = \mathrm{dom}\, g \setminus g^{-1}(\{0\})$.

(12) $\mathrm{dom}(f_1\, f_2) \setminus (f_1\, f_2)^{-1}(\{0\}) = (\mathrm{dom} f_1 \setminus {f_1}^{-1}(\{0\})) \cap (\mathrm{dom} f_2 \setminus {f_2}^{-1}(\{0\}))$.

[1] The proposition (1) has been removed.
[2] The definitions (Def. 1)–(Def. 3) have been removed.
[3] The definitions (Def. 5)–(Def. 7) have been removed.
[4] The propositions (3)–(10) have been removed.

(13) If $c \in \mathrm{dom}(\frac{1}{f})$, then $f(c) \neq 0$.

(14) $(\frac{1}{f})^{-1}(\{0\}) = \emptyset$.

(15) $|f|^{-1}(\{0\}) = f^{-1}(\{0\})$ and $(-f)^{-1}(\{0\}) = f^{-1}(\{0\})$.

(16) $\mathrm{dom}(\frac{1}{\frac{1}{f}}) = \mathrm{dom}(f \restriction \mathrm{dom}(\frac{1}{f}))$.

(17) If $r \neq 0$, then $(r\, f)^{-1}(\{0\}) = f^{-1}(\{0\})$.

(19)[5] $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(21)[6] $(f_1\, f_2)\, f_3 = f_1\, (f_2\, f_3)$.

(22) $(f_1 + f_2)\, f_3 = f_1\, f_3 + f_2\, f_3$.

(23) $f_3\, (f_1 + f_2) = f_3\, f_1 + f_3\, f_2$.

(24) $r\, (f_1\, f_2) = (r\, f_1)\, f_2$.

(25) $r\, (f_1\, f_2) = f_1\, (r\, f_2)$.

(26) $(f_1 - f_2)\, f_3 = f_1\, f_3 - f_2\, f_3$.

(27) $f_3\, f_1 - f_3\, f_2 = f_3\, (f_1 - f_2)$.

(28) $r\, (f_1 + f_2) = r\, f_1 + r\, f_2$.

(29) $(r \cdot p)\, f = r\, (p\, f)$.

(30) $r\, (f_1 - f_2) = r\, f_1 - r\, f_2$.

(31) $f_1 - f_2 = (-1)\, (f_2 - f_1)$.

(32) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(33) $1\, f = f$.

(34) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(35) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(36) $|f_1\, f_2| = |f_1|\, |f_2|$.

(37) $|r\, f| = |r|\, |f|$.

(38) $-f = (-1)\, f$.

(39) $--f = f$.

(40) $f_1 - f_2 = f_1 + -f_2$.

(41) $f_1 - -f_2 = f_1 + f_2$.

(42) $\frac{1}{\frac{1}{f}} = f \restriction \mathrm{dom}(\frac{1}{f})$.

(43) $\frac{1}{f_1\, f_2} = \frac{1}{f_1}\, \frac{1}{f_2}$.

(44) If $r \neq 0$, then $\frac{1}{r\, f} = r^{-1}\, \frac{1}{f}$.

(45) $\frac{1}{-f} = (-1)\, \frac{1}{f}$.

[5] The proposition (18) has been removed.
[6] The proposition (20) has been removed.

(46) $\frac{1}{|f|} = |\frac{1}{f}|$.

(47) $\frac{f}{g} = f\,\frac{1}{g}$.

(48) $r\,\frac{g}{f} = \frac{r\,g}{f}$.

(49) $\frac{f}{g}\,g = f\restriction\mathrm{dom}(\frac{1}{g})$.

(50) $\frac{f}{g}\,\frac{f_1}{g_1} = \frac{f\,f_1}{g\,g_1}$.

(51) $\frac{1}{\frac{f_1}{f_2}} = \frac{f_2\restriction\mathrm{dom}(\frac{1}{f_2})}{f_1}$.

(52) $g\,\frac{f_1}{f_2} = \frac{g\,f_1}{f_2}$.

(53) $\frac{g}{\frac{f_1}{f_2}} = \frac{g\,(f_2\restriction\mathrm{dom}(\frac{1}{f_2}))}{f_1}$.

(54) $-\frac{f}{g} = \frac{-f}{g}$ and $\frac{f}{-g} = -\frac{f}{g}$.

(55) $\frac{f_1}{f} + \frac{f_2}{f} = \frac{f_1+f_2}{f}$ and $\frac{f_1}{f} - \frac{f_2}{f} = \frac{f_1-f_2}{f}$.

(56) $\frac{f_1}{f} + \frac{g_1}{g} = \frac{f_1\,g+g_1\,f}{f\,g}$.

(57) $\frac{\frac{f}{g}}{\frac{f_1}{g_1}} = \frac{f\,(g_1\restriction\mathrm{dom}(\frac{1}{g_1}))}{g\,f_1}$.

(58) $\frac{f_1}{f} - \frac{g_1}{g} = \frac{f_1\,g-g_1\,f}{f\,g}$.

(59) $|\frac{f_1}{f_2}| = \frac{|f_1|}{|f_2|}$.

(60) $(f_1+f_2)\restriction X = f_1\restriction X + f_2\restriction X$ and $(f_1+f_2)\restriction X = f_1\restriction X + f_2$ and $(f_1+f_2)\restriction X = f_1 + f_2\restriction X$.

(61) $(f_1\,f_2)\restriction X = (f_1\restriction X)\,(f_2\restriction X)$ and $(f_1\,f_2)\restriction X = (f_1\restriction X)\,f_2$ and $(f_1\,f_2)\restriction X = f_1\,(f_2\restriction X)$.

(62) $(-f)\restriction X = -f\restriction X$ and $\frac{1}{f}\restriction X = \frac{1}{f\restriction X}$ and $|f|\restriction X = |f\restriction X|$.

(63) $(f_1-f_2)\restriction X = f_1\restriction X - f_2\restriction X$ and $(f_1-f_2)\restriction X = f_1\restriction X - f_2$ and $(f_1-f_2)\restriction X = f_1 - f_2\restriction X$.

(64) $\frac{f_1}{f_2}\restriction X = \frac{f_1\restriction X}{f_2\restriction X}$ and $\frac{f_1}{f_2}\restriction X = \frac{f_1\restriction X}{f_2}$ and $\frac{f_1}{f_2}\restriction X = \frac{f_1}{f_2\restriction X}$.

(65) $(r\,f)\restriction X = r\,(f\restriction X)$.

(66)(i) f_1 is total and f_2 is total iff f_1+f_2 is total,

(ii) f_1 is total and f_2 is total iff f_1-f_2 is total, and

(iii) f_1 is total and f_2 is total iff $f_1\,f_2$ is total.

(67) f is total iff $r\,f$ is total.

(68) f is total iff $-f$ is total.

(69) f is total iff $|f|$ is total.

(70) $\frac{1}{f}$ is total iff $f^{-1}(\{0\}) = \emptyset$ and f is total.

(71) f_1 is total and ${f_2}^{-1}(\{0\}) = \emptyset$ and f_2 is total iff $\frac{f_1}{f_2}$ is total.

(72) If f_1 is total and f_2 is total, then $(f_1+f_2)(c) = f_1(c)+f_2(c)$ and $(f_1-f_2)(c) = f_1(c)-f_2(c)$ and $(f_1\,f_2)(c) = f_1(c)\cdot f_2(c)$.

(73) If f is total, then $(r\,f)(c) = r \cdot f(c)$.

(74) If f is total, then $(-f)(c) = -f(c)$ and $|f|(c) = |f(c)|$.

(75) If $\frac{1}{f}$ is total, then $(\frac{1}{f})(c) = f(c)^{-1}$.

(76) If f_1 is total and $\frac{1}{f_2}$ is total, then $(\frac{f_1}{f_2})(c) = f_1(c) \cdot f_2(c)^{-1}$.

Let X, C be sets. Then $\chi_{X,C}$ is a partial function from C to $\mathbb{R}$.
The following propositions are true:

(77) $f = \chi_{X,C}$ iff $\mathrm{dom}\, f = C$ and for every c holds if $c \in X$, then $f(c) = 1$ and if $c \notin X$, then $f(c) = 0$.

(78) $\chi_{X,C}$ is total.

(79) $c \in X$ iff $\chi_{X,C}(c) = 1$.

(80) $c \notin X$ iff $\chi_{X,C}(c) = 0$.

(81) $c \in C \setminus X$ iff $\chi_{X,C}(c) = 0$.

(83)[7] $\chi_{C,C}(c) = 1$.

(84) $\chi_{X,C}(c) \neq 1$ iff $\chi_{X,C}(c) = 0$.

(85) If X misses Y, then $\chi_{X,C} + \chi_{Y,C} = \chi_{X \cup Y,C}$.

(86) $\chi_{X,C}\,\chi_{Y,C} = \chi_{X \cap Y,C}$.

Let us consider C and let us consider f, Y. We say that f is upper bounded on Y if and only if:

(Def. 9) There exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $f(c) \leq r$.

We say that f is lower bounded on Y if and only if:

(Def. 10) There exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $r \leq f(c)$.

Let us consider C and let us consider f, Y. We say that f is bounded on Y if and only if:

(Def. 11) f is upper bounded on Y and lower bounded on Y.

One can prove the following propositions:

(90)[8] f is bounded on Y iff there exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $|f(c)| \leq r$.

(91)(i) If $Y \subseteq X$ and f is upper bounded on X, then f is upper bounded on Y,

(ii) if $Y \subseteq X$ and f is lower bounded on X, then f is lower bounded on Y, and

(iii) if $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(92) If f is upper bounded on X and lower bounded on Y, then f is bounded on $X \cap Y$.

(93) If X misses $\mathrm{dom}\, f$, then f is bounded on X.

(94) If $0 = r$, then $r\,f$ is bounded on Y.

(95)(i) If f is upper bounded on Y and $0 \leq r$, then $r\,f$ is upper bounded on Y, and

(ii) if f is upper bounded on Y and $r \leq 0$, then $r\,f$ is lower bounded on Y.

[7] The proposition (82) has been removed.
[8] The propositions (87)–(89) have been removed.

(96)(i) If f is lower bounded on Y and $0 \leq r$, then $r\,f$ is lower bounded on Y, and

(ii) if f is lower bounded on Y and $r \leq 0$, then $r\,f$ is upper bounded on Y.

(97) If f is bounded on Y, then $r\,f$ is bounded on Y.

(98) $|f|$ is lower bounded on X.

(99) If f is bounded on Y, then $|f|$ is bounded on Y and $-f$ is bounded on Y.

(100)(i) If f_1 is upper bounded on X and f_2 is upper bounded on Y, then $f_1 + f_2$ is upper bounded on $X \cap Y$,

(ii) if f_1 is lower bounded on X and f_2 is lower bounded on Y, then $f_1 + f_2$ is lower bounded on $X \cap Y$, and

(iii) if f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(101) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1\,f_2$ is bounded on $X \cap Y$ and $f_1 - f_2$ is bounded on $X \cap Y$.

(102) If f is upper bounded on X and upper bounded on Y, then f is upper bounded on $X \cup Y$.

(103) If f is lower bounded on X and lower bounded on Y, then f is lower bounded on $X \cup Y$.

(104) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(105) Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$ and $f_1\,f_2$ is a constant on $X \cap Y$.

(106) If f is a constant on Y, then $p\,f$ is a constant on Y.

(107) If f is a constant on Y, then $|f|$ is a constant on Y and $-f$ is a constant on Y.

(108) If f is a constant on Y, then f is bounded on Y.

(109) If f is a constant on Y, then for every r holds $r\,f$ is bounded on Y and $-f$ is bounded on Y and $|f|$ is bounded on Y.

(110)(i) If f_1 is upper bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is upper bounded on $X \cap Y$,

(ii) if f_1 is lower bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is lower bounded on $X \cap Y$, and

(iii) if f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(111)(i) If f_1 is upper bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is upper bounded on $X \cap Y$,

(ii) if f_1 is lower bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is lower bounded on $X \cap Y$, and

(iii) if f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$ and $f_1\,f_2$ is bounded on $X \cap Y$.

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