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Inst. of Computer Science, Univ. of Białystok

Functions and Finite Sequences of Real Numbers

Jarosław Kotowicz
Warsaw University
Białystok

Summary. We define notions of fiberwise equipotent functions, non-increasing finite sequences of real numbers and new operations on finite sequences. Equivalent conditions for fiberwise equivalent functions and basic facts about new constructions are shown.

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The articles [11], [14], [12], [15], [4], [5], [3], [1], [9], [2], [10], [13], [7], [6], and [8] provide the notation and terminology for this paper.

Let F, G be binary relations. We say that F and G are fiberwise equipotent if and only if:

(Def. 1) For every set x holds $\overline{\overline{F^{-1}(\{x\})}} = \overline{\overline{G^{-1}(\{x\})}}$.

Let us notice that the predicate F and G are fiberwise equipotent is reflexive and symmetric.

We now state several propositions:

(1) For all functions F, G such that F and G are fiberwise equipotent holds $\operatorname{rng} F = \operatorname{rng} G$.

(2) Let F, G, H be functions. Suppose F and G are fiberwise equipotent and F and H are fiberwise equipotent. Then G and H are fiberwise equipotent.

(3) Let F, G be functions. Then F and G are fiberwise equipotent if and only if there exists a function H such that $\operatorname{dom} H = \operatorname{dom} F$ and $\operatorname{rng} H = \operatorname{dom} G$ and H is one-to-one and $F = G \cdot H$.

(4) For all functions F, G holds F and G are fiberwise equipotent iff for every set X holds $\overline{\overline{F^{-1}(X)}} = \overline{\overline{G^{-1}(X)}}$.

(5) Let D be a non empty set and F, G be functions. Suppose $\operatorname{rng} F \subseteq D$ and $\operatorname{rng} G \subseteq D$. Then F and G are fiberwise equipotent if and only if for every element d of D holds $\overline{\overline{F^{-1}(\{d\})}} = \overline{\overline{G^{-1}(\{d\})}}$.

(6) Let F, G be functions. Suppose $\operatorname{dom} F = \operatorname{dom} G$. Then F and G are fiberwise equipotent if and only if there exists a permutation P of $\operatorname{dom} F$ such that $F = G \cdot P$.

(7) For all functions F, G such that F and G are fiberwise equipotent holds $\overline{\overline{\operatorname{dom} F}} = \overline{\overline{\operatorname{dom} G}}$.

Let F be a finite function and let A be a set. Observe that $F^{-1}(A)$ is finite.

Next we state several propositions:

(9)[1] Let F, G be finite functions. Then F and G are fiberwise equipotent if and only if for every set X holds $\operatorname{card}(F^{-1}(X)) = \operatorname{card}(G^{-1}(X))$.

[1] The proposition (8) has been removed.

(10) For all finite functions F, G such that F and G are fiberwise equipotent holds $\operatorname{card}\operatorname{dom} F = \operatorname{card}\operatorname{dom} G$.

(11) Let D be a non empty set and F, G be finite functions. Suppose $\operatorname{rng} F \subseteq D$ and $\operatorname{rng} G \subseteq D$. Then F and G are fiberwise equipotent if and only if for every element d of D holds $\operatorname{card}(F^{-1}(\{d\})) = \operatorname{card}(G^{-1}(\{d\}))$.

(13)[2] Let f, g be finite sequences. Then f and g are fiberwise equipotent if and only if for every set X holds $\operatorname{card}(f^{-1}(X)) = \operatorname{card}(g^{-1}(X))$.

(14) Let f, g, h be finite sequences. Then f and g are fiberwise equipotent if and only if $f \frown h$ and $g \frown h$ are fiberwise equipotent.

(15) For all finite sequences f, g holds $f \frown g$ and $g \frown f$ are fiberwise equipotent.

(16) For all finite sequences f, g such that f and g are fiberwise equipotent holds $\operatorname{len} f = \operatorname{len} g$ and $\operatorname{dom} f = \operatorname{dom} g$.

(17) Let f, g be finite sequences. Then f and g are fiberwise equipotent if and only if there exists a permutation P of $\operatorname{dom} g$ such that $f = g \cdot P$.

Let F be a function and let X be a finite set. One can verify that $F{\restriction}X$ is finite and function-like. Next we state the proposition

(18) Let F be a function and X be a finite set. Then there exists a finite sequence f such that $F{\restriction}X$ and f are fiberwise equipotent.

Let D be a set, let f be a finite sequence of elements of D, and let n be a natural number. The functor $f_{\downharpoonleft n}$ yields a finite sequence of elements of D and is defined as follows:

(Def. 2)(i) $\operatorname{len}(f_{\downharpoonleft n}) = \operatorname{len} f - n$ and for every natural number m such that $m \in \operatorname{dom}(f_{\downharpoonleft n})$ holds $f_{\downharpoonleft n}(m) = f(m+n)$ if $n \leq \operatorname{len} f$,

(ii) $f_{\downharpoonleft n} = \varepsilon_D$, otherwise.

Next we state four propositions:

(19) Let D be a non empty set, f be a finite sequence of elements of D, and n, m be natural numbers. If $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} n$, then $(f{\restriction}n)(m) = f(m)$ and $m \in \operatorname{dom} f$.

(20) Let D be a non empty set, f be a finite sequence of elements of D, n be a natural number, and x be a set. If $\operatorname{len} f = n+1$ and $x = f(n+1)$, then $f = (f{\restriction}n) \frown \langle x \rangle$.

(21) Let D be a non empty set, f be a finite sequence of elements of D, and n be a natural number. Then $(f{\restriction}n) \frown (f_{\downharpoonleft n}) = f$.

(22) For all finite sequences R_1, R_2 of elements of $\mathbb{R}$ such that R_1 and R_2 are fiberwise equipotent holds $\sum R_1 = \sum R_2$.

Let R be a finite sequence of elements of $\mathbb{R}$. The functor $\operatorname{MIM}(R)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the conditions (Def. 3).

(Def. 3)(i) $\operatorname{len}\operatorname{MIM}(R) = \operatorname{len} R$,

(ii) $(\operatorname{MIM}(R))(\operatorname{len}\operatorname{MIM}(R)) = R(\operatorname{len} R)$, and

(iii) for every natural number n such that $1 \leq n$ and $n \leq \operatorname{len}\operatorname{MIM}(R) - 1$ holds $(\operatorname{MIM}(R))(n) = R(n) - R(n+1)$.

We now state several propositions:

(23) Let R be a finite sequence of elements of $\mathbb{R}$, r be a real number, and n be a natural number. If $\operatorname{len} R = n+2$ and $R(n+1) = r$, then $\operatorname{MIM}(R{\restriction}(n+1)) = (\operatorname{MIM}(R){\restriction}n) \frown \langle r \rangle$.

[2] The proposition (12) has been removed.

(24) Let R be a finite sequence of elements of $\mathbb{R}$, r, s be real numbers, and n be a natural number. If $\mathrm{len}\,R = n+2$ and $R(n+1) = r$ and $R(n+2) = s$, then $\mathrm{MIM}(R) = (\mathrm{MIM}(R){\restriction}n) ^\frown \langle r-s, s \rangle$.

(25) $\mathrm{MIM}(\varepsilon_{\mathbb{R}}) = \varepsilon_{\mathbb{R}}$.

(26) For every real number r holds $\mathrm{MIM}(\langle r \rangle) = \langle r \rangle$.

(27) For all real numbers r, s holds $\mathrm{MIM}(\langle r, s \rangle) = \langle r-s, s \rangle$.

(28) For every finite sequence R of elements of $\mathbb{R}$ and for every natural number n holds $(\mathrm{MIM}(R))_{\lfloor n} = \mathrm{MIM}(R_{\lfloor n})$.

(29) For every finite sequence R of elements of $\mathbb{R}$ such that $\mathrm{len}\,R \neq 0$ holds $\sum \mathrm{MIM}(R) = R(1)$.

(30) Let R be a finite sequence of elements of $\mathbb{R}$ and n be a natural number. If $1 \leq n$ and $n < \mathrm{len}\,R$, then $\sum \mathrm{MIM}(R_{\lfloor n}) = R(n+1)$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is non-increasing if and only if:

(Def. 4) For every natural number n such that $n \in \mathrm{dom}\,I_1$ and $n+1 \in \mathrm{dom}\,I_1$ holds $I_1(n) \geq I_1(n+1)$.

One can check that there exists a finite sequence of elements of $\mathbb{R}$ which is non-increasing.

We now state several propositions:

(31) For every finite sequence R of elements of $\mathbb{R}$ such that $\mathrm{len}\,R = 0$ or $\mathrm{len}\,R = 1$ holds R is non-increasing.

(32) Let R be a finite sequence of elements of $\mathbb{R}$. Then R is non-increasing if and only if for all natural numbers n, m such that $n \in \mathrm{dom}\,R$ and $m \in \mathrm{dom}\,R$ and $n < m$ holds $R(n) \geq R(m)$.

(33) Let R be a non-increasing finite sequence of elements of $\mathbb{R}$ and n be a natural number. Then $R{\restriction}n$ is a non-increasing finite sequence of elements of $\mathbb{R}$.

(34) Let R be a non-increasing finite sequence of elements of $\mathbb{R}$ and n be a natural number. Then $R_{\lfloor n}$ is a non-increasing finite sequence of elements of $\mathbb{R}$.

(35) Let R be a finite sequence of elements of $\mathbb{R}$. Then there exists a non-increasing finite sequence R_1 of elements of $\mathbb{R}$ such that R and R_1 are fiberwise equipotent.

(36) Let R_1, R_2 be non-increasing finite sequences of elements of $\mathbb{R}$. If R_1 and R_2 are fiberwise equipotent, then $R_1 = R_2$.

(37) For every finite sequence R of elements of $\mathbb{R}$ and for all real numbers r, s such that $r \neq 0$ holds $R^{-1}(\{\frac{s}{r}\}) = (r \cdot R)^{-1}(\{s\})$.

(38) For every finite sequence R of elements of $\mathbb{R}$ holds $(0 \cdot R)^{-1}(\{0\}) = \mathrm{dom}\,R$.

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