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Relations Defined on Sets

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Summary. The article includes theorems concerning properties of relations defined as a subset of the Cartesian product of two sets (mode Relation of X,Y where X,Y are sets). Some notions, introduced in [4] such as domain, codomain, field of a relation, composition of relations, image and inverse image of a set under a relation are redefined.

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The articles [2], [1], [3], and [4] provide the notation and terminology for this paper.

We adopt the following convention: A, B, X, X_1, Y, Y_1, Y_2, Z denote sets and a, x, y denote sets.

Let us consider X, Y. Relation between X and Y is defined by:

(Def. 1) It $\subseteq [:X, Y:]$.

Let us consider X, Y. We see that the relation between X and Y is a subset of $[:X, Y:]$.

Let us consider X, Y. Note that every subset of $[:X, Y:]$ is relation-like.

In the sequel P, R denote relations between X and Y.

One can prove the following propositions:

(4)[1] If $A \subseteq R$, then A is a relation between X and Y.

(6)[2] If $a \in R$, then there exist x, y such that $a = \langle x, y \rangle$ and $x \in X$ and $y \in Y$.

(8)[3] If $x \in X$ and $y \in Y$, then $\{\langle x, y \rangle\}$ is a relation between X and Y.

(9) For every binary relation R such that $\mathrm{dom}\, R \subseteq X$ holds R is a relation between X and $\mathrm{rng}\, R$.

(10) For every binary relation R such that $\mathrm{rng}\, R \subseteq Y$ holds R is a relation between $\mathrm{dom}\, R$ and Y.

(11) For every binary relation R such that $\mathrm{dom}\, R \subseteq X$ and $\mathrm{rng}\, R \subseteq Y$ holds R is a relation between X and Y.

(12) $\mathrm{dom}\, R \subseteq X$ and $\mathrm{rng}\, R \subseteq Y$.

(13) If $\mathrm{dom}\, R \subseteq X_1$, then R is a relation between X_1 and Y.

(14) If $\mathrm{rng}\, R \subseteq Y_1$, then R is a relation between X and Y_1.

(15) If $X \subseteq X_1$, then R is a relation between X_1 and Y.

(16) If $Y \subseteq Y_1$, then R is a relation between X and Y_1.

[1] The propositions (1)–(3) have been removed.
[2] The proposition (5) has been removed.
[3] The proposition (7) has been removed.

(17) If $X \subseteq X_1$ and $Y \subseteq Y_1$, then R is a relation between X_1 and Y_1.

Let us consider X, Y, P, R. Then $P \cup R$ is a relation between X and Y. Then $P \cap R$ is a relation between X and Y. Then $P \setminus R$ is a relation between X and Y.

Let us consider X, Y, R. Then $\mathrm{dom} R$ is a subset of X. Then $\mathrm{rng} R$ is a subset of Y.

The following propositions are true:

(19)[4] $\mathrm{field} R \subseteq X \cup Y$.

(22)[5] For every x such that $x \in X$ there exists y such that $\langle x, y \rangle \in R$ iff $\mathrm{dom} R = X$.

(23) For every y such that $y \in Y$ there exists x such that $\langle x, y \rangle \in R$ iff $\mathrm{rng} R = Y$.

Let us consider X, Y, R. Then $R^{\smile}$ is a relation between Y and X.

Let us consider X, Y_1, Y_2, Z, let P be a relation between X and Y_1, and let R be a relation between Y_2 and Z. Then $P \cdot R$ is a relation between X and Z.

Next we state several propositions:

(24) $\mathrm{dom}(R^{\smile}) = \mathrm{rng} R$ and $\mathrm{rng}(R^{\smile}) = \mathrm{dom} R$.

(25) $\emptyset$ is a relation between X and Y.

(26) If R is a relation between $\emptyset$ and Y, then $R = \emptyset$.

(27) If R is a relation between X and $\emptyset$, then $R = \emptyset$.

(28) $\mathrm{id}_X \subseteq [: X, X :]$.

(29) id_X is a relation between X and X.

(30) If $\mathrm{id}_A \subseteq R$, then $A \subseteq \mathrm{dom} R$ and $A \subseteq \mathrm{rng} R$.

(31) If $\mathrm{id}_X \subseteq R$, then $X = \mathrm{dom} R$ and $X \subseteq \mathrm{rng} R$.

(32) If $\mathrm{id}_Y \subseteq R$, then $Y \subseteq \mathrm{dom} R$ and $Y = \mathrm{rng} R$.

Let us consider X, Y, R, A. Then $R \restriction A$ is a relation between X and Y.

Let us consider X, Y, B, R. Then $B \restriction R$ is a relation between X and Y.

Next we state four propositions:

(33) $R \restriction X_1$ is a relation between X_1 and Y.

(34) If $X \subseteq X_1$, then $R \restriction X_1 = R$.

(35) $Y_1 \restriction R$ is a relation between X and Y_1.

(36) If $Y \subseteq Y_1$, then $Y_1 \restriction R = R$.

Let us consider X, Y, R, A. Then $R^{\circ} A$ is a subset of Y. Then $R^{-1}(A)$ is a subset of X.

Next we state two propositions:

(38)[6] $R^{\circ} X = \mathrm{rng} R$ and $R^{-1}(Y) = \mathrm{dom} R$.

(39) $R^{\circ} R^{-1}(Y) = \mathrm{rng} R$ and $R^{-1}(R^{\circ} X) = \mathrm{dom} R$.

The scheme *Rel On Set Ex* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a relation R between $\mathcal{A}$ and $\mathcal{B}$ such that for all x, y holds $\langle x, y \rangle \in R$ iff $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$

for all values of the parameters.

Let us consider X. A binary relation on X is a relation between X and X.

In the sequel R is a binary relation on X.

The following proposition is true

[4] The proposition (18) has been removed.

[5] The propositions (20) and (21) have been removed.

[6] The proposition (37) has been removed.

(45)[7] $R \cdot \mathrm{id}_X = R$ and $\mathrm{id}_X \cdot R = R$.

For simplicity, we adopt the following rules: D, D_1, D_2, E, F denote non empty sets, R denotes a relation between D and E, x denotes an element of D, and y denotes an element of E.

We now state several propositions:

(46) $\mathrm{id}_D \neq \emptyset$.

(47) For every element x of D holds $x \in \mathrm{dom} R$ iff there exists an element y of E such that $\langle x, y\rangle \in R$.

(48) For every element y of E holds $y \in \mathrm{rng} R$ iff there exists an element x of D such that $\langle x, y\rangle \in R$.

(49) For every element x of D such that $x \in \mathrm{dom} R$ there exists an element y of E such that $y \in \mathrm{rng} R$.

(50) For every element y of E such that $y \in \mathrm{rng} R$ there exists an element x of D such that $x \in \mathrm{dom} R$.

(51) Let P be a relation between D and E, R be a relation between E and F, x be an element of D, and z be an element of F. Then $\langle x, z\rangle \in P \cdot R$ if and only if there exists an element y of E such that $\langle x, y\rangle \in P$ and $\langle y, z\rangle \in R$.

(52) $y \in R^{\circ} D_1$ iff there exists an element x of D such that $\langle x, y\rangle \in R$ and $x \in D_1$.

(53) $x \in R^{-1}(D_2)$ iff there exists an element y of E such that $\langle x, y\rangle \in R$ and $y \in D_2$.

The scheme *Rel On Dom Ex* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a relation R between $\mathcal{A}$ and $\mathcal{B}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in R$ if and only if $\mathcal{P}[x, y]$

for all values of the parameters.

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[7] The propositions (40)–(44) have been removed.