JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Properties of Binary Relations[1]

Edmund Woronowicz
Warsaw University
Białystok

Anna Zalewska
Warsaw University
Białystok

Summary. The paper contains definitions of some properties of binary relations: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, strong connectedness, and transitivity. Basic theorems relating the above mentioned notions are given.

MML Identifier: RELAT_2.

WWW: http://mizar.org/JFM/Vol1/relat_2.html

The articles [2], [1], and [3] provide the notation and terminology for this paper.

We follow the rules: X denotes a set, x, y, z denote sets, and P, R denote binary relations.

Let us consider R, X. We say that R is reflexive in X if and only if:

(Def. 1) If $x \in X$, then $\langle x, x \rangle \in R$.

We say that R is irreflexive in X if and only if:

(Def. 2) If $x \in X$, then $\langle x, x \rangle \notin R$.

We say that R is symmetric in X if and only if:

(Def. 3) If $x \in X$ and $y \in X$ and $\langle x, y \rangle \in R$, then $\langle y, x \rangle \in R$.

We say that R is antisymmetric in X if and only if:

(Def. 4) If $x \in X$ and $y \in X$ and $\langle x, y \rangle \in R$ and $\langle y, x \rangle \in R$, then $x = y$.

We say that R is asymmetric in X if and only if:

(Def. 5) If $x \in X$ and $y \in X$ and $\langle x, y \rangle \in R$, then $\langle y, x \rangle \notin R$.

We say that R is connected in X if and only if:

(Def. 6) If $x \in X$ and $y \in X$ and $x \neq y$, then $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$.

We say that R is strongly connected in X if and only if:

(Def. 7) If $x \in X$ and $y \in X$, then $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$.

We say that R is transitive in X if and only if:

(Def. 8) If $x \in X$ and $y \in X$ and $z \in X$ and $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$, then $\langle x, z \rangle \in R$.

Let us consider R. We say that R is reflexive if and only if:

(Def. 9) R is reflexive in field R.

[1]Supported by RPBP.III-24.C1.

We say that R is irreflexive if and only if:

(Def. 10) R is irreflexive in $\operatorname{field} R$.

We say that R is symmetric if and only if:

(Def. 11) R is symmetric in $\operatorname{field} R$.

We say that R is antisymmetric if and only if:

(Def. 12) R is antisymmetric in $\operatorname{field} R$.

We say that R is asymmetric if and only if:

(Def. 13) R is asymmetric in $\operatorname{field} R$.

We say that R is connected if and only if:

(Def. 14) R is connected in $\operatorname{field} R$.

We say that R is strongly connected if and only if:

(Def. 15) R is strongly connected in $\operatorname{field} R$.

We say that R is transitive if and only if:

(Def. 16) R is transitive in $\operatorname{field} R$.

One can prove the following propositions:

(17)[1] R is reflexive iff $\mathrm{id}_{\operatorname{field} R} \subseteq R$.

(18) R is irreflexive iff $\mathrm{id}_{\operatorname{field} R}$ misses R.

(19) R is antisymmetric in X iff $R \setminus \mathrm{id}_X$ is asymmetric in X.

(20) If R is asymmetric in X, then $R \cup \mathrm{id}_X$ is antisymmetric in X.

(21) If R is antisymmetric in X, then $R \setminus \mathrm{id}_X$ is asymmetric in X.

(22) If R is symmetric and transitive, then R is reflexive.

(23) id_X is symmetric and id_X is transitive.

(24) id_X is antisymmetric and id_X is reflexive.

(25) If R is irreflexive and transitive, then R is asymmetric.

(26) If R is asymmetric, then R is irreflexive and antisymmetric.

(27) If R is reflexive, then $R^{\smile}$ is reflexive.

(28) If R is irreflexive, then $R^{\smile}$ is irreflexive.

(29) If R is reflexive, then $\operatorname{dom} R = \operatorname{dom}(R^{\smile})$ and $\operatorname{rng} R = \operatorname{rng}(R^{\smile})$.

(30) R is symmetric iff $R = R^{\smile}$.

(31) If P is reflexive and R is reflexive, then $P \cup R$ is reflexive and $P \cap R$ is reflexive.

(32) If P is irreflexive and R is irreflexive, then $P \cup R$ is irreflexive and $P \cap R$ is irreflexive.

(33) If P is irreflexive, then $P \setminus R$ is irreflexive.

(34) If R is symmetric, then $R^{\smile}$ is symmetric.

[1] The propositions (1)–(16) have been removed.

(35) If P is symmetric and R is symmetric, then $P \cup R$ is symmetric and $P \cap R$ is symmetric and $P \setminus R$ is symmetric.

(36) If R is asymmetric, then $R^{\smile}$ is asymmetric.

(37) If P is asymmetric and R is asymmetric, then $P \cap R$ is asymmetric.

(38) If P is asymmetric, then $P \setminus R$ is asymmetric.

(39) R is antisymmetric iff $R \cap R^{\smile} \subseteq \mathrm{id}_{\mathrm{dom}\,R}$.

(40) If R is antisymmetric, then $R^{\smile}$ is antisymmetric.

(41) If P is antisymmetric, then $P \cap R$ is antisymmetric and $P \setminus R$ is antisymmetric.

(42) If R is transitive, then $R^{\smile}$ is transitive.

(43) If P is transitive and R is transitive, then $P \cap R$ is transitive.

(44) R is transitive iff $R \cdot R \subseteq R$.

(45) R is connected iff $[:\mathrm{field}\,R, \mathrm{field}\,R:] \setminus \mathrm{id}_{\mathrm{field}\,R} \subseteq R \cup R^{\smile}$.

(46) If R is strongly connected, then R is connected and reflexive.

(47) R is strongly connected iff $[:\mathrm{field}\,R, \mathrm{field}\,R:] = R \cup R^{\smile}$.

REFERENCES

[1] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[2] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[3] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received March 15, 1989

Published January 2, 2004