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Several Properties of Fields. Field Theory

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Summary. The article includes a continuation of the paper [1]. Some simple theorems concerning basic properties of a field are proved.

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The articles [5], [4], [6], [2], [3], and [1] provide the notation and terminology for this paper.

The following propositions are true:

(1) For every field F holds $-_F(\mathbf{0}_F) = \mathbf{0}_F$.

(2) For every field F holds $({}^{-1}_{F})(\mathbf{1}_F) = \mathbf{1}_F$.

(3) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, -_F(b)\rangle)) = +_F(\langle b, -_F(a)\rangle)$.

(4) For every field F and for all elements a, b of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $({}^{-1}_{F})(\cdot_F(\langle a, ({}^{-1}_{F})(b)\rangle)) = \cdot_F(\langle b, ({}^{-1}_{F})(a)\rangle)$.

(5) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, b\rangle)) = +_F(\langle -_F(a), -_F(b)\rangle)$.

(6) For every field F and for all elements a, b of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $({}^{-1}_{F})(\cdot_F(\langle a, b\rangle)) = \cdot_F(\langle ({}^{-1}_{F})(a), ({}^{-1}_{F})(b)\rangle)$.

(7) Let F be a field and a, b, c, d be elements of the support of F. Then $+_F(\langle a, -_F(b)\rangle) = +_F(\langle c, -_F(d)\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(8) Let F be a field, a, c be elements of the support of F, and b, d be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, ({}^{-1}_{F})(b)\rangle) = \cdot_F(\langle c, ({}^{-1}_{F})(d)\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(9) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle) = \mathbf{0}_F$ iff $a = \mathbf{0}_F$ or $b = \mathbf{0}_F$.

(10) Let F be a field, a, b be elements of the support of F, and c, d be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle \cdot_F(\langle a, ({}^{-1}_{F})(c)\rangle), \cdot_F(\langle b, ({}^{-1}_{F})(d)\rangle)\rangle) = \cdot_F(\langle \cdot_F(\langle a, b\rangle), ({}^{-1}_{F})(\cdot_F(\langle c, d\rangle))\rangle)$.

(11) Let F be a field, a, b be elements of the support of F, and c, d be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $+_F(\langle \cdot_F(\langle a, ({}^{-1}_{F})(c)\rangle), \cdot_F(\langle b, ({}^{-1}_{F})(d)\rangle)\rangle) = \cdot_F(\langle +_F(\langle \cdot_F(\langle a, d\rangle), \cdot_F(\langle b, c\rangle)\rangle), ({}^{-1}_{F})(\cdot_F(\langle c, d\rangle))\rangle)$.

Let F be a field. The functor $\operatorname{osf} F$ yielding a binary operation on the support of F is defined by:

(Def. 1) For all elements x, y of the support of F holds $(\operatorname{osf} F)(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

Next we state a number of propositions:

(14)[1] For every field F and for every element x of the support of F holds $(\mathrm{osf}F)(\langle x, x\rangle) = \mathbf{0}_F$.

(15) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle a, (\mathrm{osf}F)(\langle b, c\rangle)\rangle) = (\mathrm{osf}F)(\langle \cdot_F(\langle a, b\rangle), \cdot_F(\langle a, c\rangle)\rangle)$.

(16) Let F be a field and a, b be elements of the support of F. Then $(\mathrm{osf}F)(\langle a, b\rangle)$ is an element of the support of F.

(17) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle (\mathrm{osf}F)(\langle a, b\rangle), c\rangle) = (\mathrm{osf}F)(\langle \cdot_F(\langle a, c\rangle), \cdot_F(\langle b, c\rangle)\rangle)$.

(18) For every field F and for all elements a, b of the support of F holds $(\mathrm{osf}F)(\langle a, b\rangle) = -_F((\mathrm{osf}F)(\langle b, a\rangle))$.

(19) For every field F and for all elements a, b of the support of F holds $(\mathrm{osf}F)(\langle -_F(a), b\rangle) = -_F(+_F(\langle a, b\rangle))$.

(20) Let F be a field and a, b, c, d be elements of the support of F. Then $(\mathrm{osf}F)(\langle a, b\rangle) = (\mathrm{osf}F)(\langle c, d\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(21) For every field F and for every element a of the support of F holds $(\mathrm{osf}F)(\langle \mathbf{0}_F, a\rangle) = -_F(a)$.

(22) For every field F and for every element a of the support of F holds $(\mathrm{osf}F)(\langle a, \mathbf{0}_F\rangle) = a$.

(23) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ iff $(\mathrm{osf}F)(\langle c, a\rangle) = b$.

(24) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ iff $(\mathrm{osf}F)(\langle c, b\rangle) = a$.

(25) For every field F and for all elements a, b, c of the support of F holds $(\mathrm{osf}F)(\langle a, (\mathrm{osf}F)(\langle b, c\rangle)\rangle) = +_F(\langle (\mathrm{osf}F)(\langle a, b\rangle), c\rangle)$.

(26) For every field F and for all elements a, b, c of the support of F holds $(\mathrm{osf}F)(\langle a, +_F(\langle b, c\rangle)\rangle) = (\mathrm{osf}F)(\langle (\mathrm{osf}F)(\langle a, b\rangle), c\rangle)$.

Let F be a field. The functor $\mathrm{ovf}F$ yields a function from $[\!:$ the support of F, (the support of $F) \setminus \{\mathbf{0}_F\} :\!]$ into the support of F and is defined by the condition (Def. 2).

(Def. 2) Let x be an element of the support of F and y be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle x, y\rangle) = \cdot_F(\langle x, (\overset{-1}{{}_F})(y)\rangle)$.

We now state a number of propositions:

(29)[2] For every field F and for every element x of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle x, x\rangle) = \mathbf{1}_F$.

(30) Let F be a field, a be an element of the support of F, and b be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, b\rangle)$ is an element of the support of F.

(31) Let F be a field, a, b be elements of the support of F, and c be an element of (the support of $F) \setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, (\mathrm{ovf}F)(\langle b, c\rangle)\rangle) = (\mathrm{ovf}F)(\langle \cdot_F(\langle a, b\rangle), c\rangle)$.

(32) For every field F and for every element a of (the support of $F) \setminus \{\mathbf{0}_F\}$ holds $\cdot_F(\langle a, (\mathrm{ovf}F)(\langle \mathbf{1}_F, a\rangle)\rangle) = \mathbf{1}_F$ and $\cdot_F(\langle (\mathrm{ovf}F)(\langle \mathbf{1}_F, a\rangle), a\rangle) = \mathbf{1}_F$.

[1] The propositions (12) and (13) have been removed.

[2] The propositions (27) and (28) have been removed.

(35)[3] For every field F and for all elements a, b of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle a, b\rangle) = (\overset{-1}{F})((\mathrm{ovf}F)(\langle b, a\rangle))$.

(36) For every field F and for all elements a, b of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle(\overset{-1}{F})(a), b\rangle) = (\overset{-1}{F})(\cdot_F(\langle a, b\rangle))$.

(37) Let F be a field, a, c be elements of the support of F, and b, d be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, b\rangle) = (\mathrm{ovf}F)(\langle c, d\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(38) For every field F and for every element a of (the support of F) $\setminus \{\mathbf{0}_F\}$ holds $(\mathrm{ovf}F)(\langle \mathbf{1}_F, a\rangle) = (\overset{-1}{F})(a)$.

(39) For every field F and for every element a of the support of F holds $(\mathrm{ovf}F)(\langle a, \mathbf{1}_F\rangle) = a$.

(40) Let F be a field, a be an element of (the support of F) $\setminus \{\mathbf{0}_F\}$, and b, c be elements of the support of F. Then $\cdot_F(\langle a, b\rangle) = c$ if and only if $(\mathrm{ovf}F)(\langle c, a\rangle) = b$.

(41) Let F be a field, a, c be elements of the support of F, and b be an element of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $\cdot_F(\langle a, b\rangle) = c$ if and only if $(\mathrm{ovf}F)(\langle c, b\rangle) = a$.

(42) Let F be a field, a be an element of the support of F, and b, c be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, (\mathrm{ovf}F)(\langle b, c\rangle)\rangle) = \cdot_F(\langle(\mathrm{ovf}F)(\langle a, b\rangle), c\rangle)$.

(43) Let F be a field, a be an element of the support of F, and b, c be elements of (the support of F) $\setminus \{\mathbf{0}_F\}$. Then $(\mathrm{ovf}F)(\langle a, \cdot_F(\langle b, c\rangle)\rangle) = (\mathrm{ovf}F)(\langle(\mathrm{ovf}F)(\langle a, b\rangle), c\rangle)$.

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[3] The propositions (33) and (34) have been removed.