JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Lattice of Real Numbers. The Lattice of Real Functions

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Summary. A proof of the fact, that $\langle \mathbb{R}, \max, \min \rangle$ is a lattice (real lattice). Some basic properties (real lattice is distributive and modular) of it are proved. The same is done for the set $\mathbb{R}^A$ with operations: $\max(f(A))$ and $\min(f(A))$, where $\mathbb{R}^A$ means the set of all functions from A (being non-empty set) to $\mathbb{R}$, f is just such a function.

MML Identifier: `REAL_LAT`.

WWW: `http://mizar.org/JFM/Vol2/real_lat.html`

The articles [7], [5], [6], [9], [1], [3], [8], [2], and [4] provide the notation and terminology for this paper.

In this paper x, y denote real numbers.

The binary operation $\min_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:

(Def. 1) $\min_{\mathbb{R}}(x, y) = \min(x, y)$.

The binary operation $\max_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:

(Def. 2) $\max_{\mathbb{R}}(x, y) = \max(x, y)$.

The strict lattice structure $\mathbb{R}_{\mathrm{L}}$ is defined as follows:

(Def. 4)[1] $\mathbb{R}_{\mathrm{L}} = \langle \mathbb{R}, \max_{\mathbb{R}}, \min_{\mathbb{R}} \rangle$.

Let us mention that every element of $\mathbb{R}_{\mathrm{L}}$ is real.

One can verify that $\mathbb{R}_{\mathrm{L}}$ is non empty.

Let us mention that $\mathbb{R}_{\mathrm{L}}$ is lattice-like.

In the sequel p, q, r are elements of $\mathbb{R}_{\mathrm{L}}$.

We now state several propositions:

(8)[2] $\max_{\mathbb{R}}(p, q) = \max_{\mathbb{R}}(q, p)$.

(9) $\min_{\mathbb{R}}(p, q) = \min_{\mathbb{R}}(q, p)$.

(10)(i) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, r), p)$,

(ii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), r)$,

(iii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), r)$,

(iv) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, p), q)$,

(v) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, q), p)$, and

(vi) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, r), q)$.

[1] The definition (Def. 3) has been removed.
[2] The propositions (1)–(7) have been removed.

(11)(i) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, r), p)$,

(ii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), r)$,

(iii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), r)$,

(iv) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, p), q)$,

(v) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, q), p)$, and

(vi) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, r), q)$.

(12) $\max_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), q) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(p, q)) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(q, p)) = q$ and $\max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), q) = q$.

(13) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(q, p)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), q) = q$ and $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, q)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), q) = q$.

(14) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, r)) = \max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), \min_{\mathbb{R}}(q, r))$.

One can verify that $\mathbb{R}_{\mathrm{L}}$ is distributive.

In the sequel A is a non empty set and f, g, h are elements of $\mathbb{R}^A$.

Let us consider A. The functor $\mathbf{max}_{\mathbb{R}^A}$ yielding a binary operation on $\mathbb{R}^A$ is defined by:

(Def. 5) $\mathbf{max}_{\mathbb{R}^A}(f, g) = (\max_{\mathbb{R}})^{\circ}(f, g)$.

The functor $\mathbf{min}_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:

(Def. 6) $\mathbf{min}_{\mathbb{R}^A}(f, g) = (\min_{\mathbb{R}})^{\circ}(f, g)$.

The following propositions are true:

(20)[3] $\mathbf{max}_{\mathbb{R}^A}(f, g) = \mathbf{max}_{\mathbb{R}^A}(g, f)$.

(21) $\mathbf{min}_{\mathbb{R}^A}(f, g) = \mathbf{min}_{\mathbb{R}^A}(g, f)$.

(22) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(f, g), h) = \mathbf{max}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(g, h))$.

(23) $\mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(f, g), h) = \mathbf{min}_{\mathbb{R}^A}(f, \mathbf{min}_{\mathbb{R}^A}(g, h))$.

(24) $\mathbf{max}_{\mathbb{R}^A}(f, \mathbf{min}_{\mathbb{R}^A}(f, g)) = f$.

(25) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(f, g), f) = f$.

(26) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(g, f), f) = f$.

(27) $\mathbf{max}_{\mathbb{R}^A}(f, \mathbf{min}_{\mathbb{R}^A}(g, f)) = f$.

(28) $\mathbf{min}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(f, g)) = f$.

(29) $\mathbf{min}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(g, f)) = f$.

(30) $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(g, f), f) = f$.

(31) $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(f, g), f) = f$.

(32) $\mathbf{min}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(g, h)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(f, g), \mathbf{min}_{\mathbb{R}^A}(f, h))$.

Let us consider A and let x be an element of $\langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$. The functor ${}^{@}x$ yielding an element of $\mathbb{R}^A$ is defined as follows:

(Def. 9)[4] ${}^{@}x = x$.

Let us consider A. The functor $\mathbb{R}^A_{\mathrm{L}}$ yields a strict lattice and is defined by:

[3] The propositions (15)–(19) have been removed.

[4] The definitions (Def. 7) and (Def. 8) have been removed.

(Def. 10) $\mathbb{R}^{A}_{\mathrm{L}} = \langle \mathbb{R}^{A}, \mathbf{max}_{\mathbb{R}^{A}}, \mathbf{min}_{\mathbb{R}^{A}} \rangle$.

In the sequel p, q, r are elements of $\mathbb{R}^{A}_{\mathrm{L}}$.

One can prove the following propositions:

(40)[5] $\mathbf{max}_{\mathbb{R}^{A}}(p, q) = \mathbf{max}_{\mathbb{R}^{A}}(q, p)$.

(41) $\mathbf{min}_{\mathbb{R}^{A}}(p, q) = \mathbf{min}_{\mathbb{R}^{A}}(q, p)$.

(42)(i) $\mathbf{max}_{\mathbb{R}^{A}}(p, \mathbf{max}_{\mathbb{R}^{A}}(q, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(q, r), p)$,

(ii) $\mathbf{max}_{\mathbb{R}^{A}}(p, \mathbf{max}_{\mathbb{R}^{A}}(q, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(p, q), r)$,

(iii) $\mathbf{max}_{\mathbb{R}^{A}}(p, \mathbf{max}_{\mathbb{R}^{A}}(q, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(q, p), r)$,

(iv) $\mathbf{max}_{\mathbb{R}^{A}}(p, \mathbf{max}_{\mathbb{R}^{A}}(q, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(r, p), q)$,

(v) $\mathbf{max}_{\mathbb{R}^{A}}(p, \mathbf{max}_{\mathbb{R}^{A}}(q, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(r, q), p)$, and

(vi) $\mathbf{max}_{\mathbb{R}^{A}}(p, \mathbf{max}_{\mathbb{R}^{A}}(q, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(p, r), q)$.

(43)(i) $\mathbf{min}_{\mathbb{R}^{A}}(p, \mathbf{min}_{\mathbb{R}^{A}}(q, r)) = \mathbf{min}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(q, r), p)$,

(ii) $\mathbf{min}_{\mathbb{R}^{A}}(p, \mathbf{min}_{\mathbb{R}^{A}}(q, r)) = \mathbf{min}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(p, q), r)$,

(iii) $\mathbf{min}_{\mathbb{R}^{A}}(p, \mathbf{min}_{\mathbb{R}^{A}}(q, r)) = \mathbf{min}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(q, p), r)$,

(iv) $\mathbf{min}_{\mathbb{R}^{A}}(p, \mathbf{min}_{\mathbb{R}^{A}}(q, r)) = \mathbf{min}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(r, p), q)$,

(v) $\mathbf{min}_{\mathbb{R}^{A}}(p, \mathbf{min}_{\mathbb{R}^{A}}(q, r)) = \mathbf{min}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(r, q), p)$, and

(vi) $\mathbf{min}_{\mathbb{R}^{A}}(p, \mathbf{min}_{\mathbb{R}^{A}}(q, r)) = \mathbf{min}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(p, r), q)$.

(44) $\mathbf{max}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(p, q), q) = q$ and $\mathbf{max}_{\mathbb{R}^{A}}(q, \mathbf{min}_{\mathbb{R}^{A}}(p, q)) = q$ and $\mathbf{max}_{\mathbb{R}^{A}}(q, \mathbf{min}_{\mathbb{R}^{A}}(q, p)) = q$ and $\mathbf{max}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(q, p), q) = q$.

(45) $\mathbf{min}_{\mathbb{R}^{A}}(q, \mathbf{max}_{\mathbb{R}^{A}}(q, p)) = q$ and $\mathbf{min}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(p, q), q) = q$ and $\mathbf{min}_{\mathbb{R}^{A}}(q, \mathbf{max}_{\mathbb{R}^{A}}(p, q)) = q$ and $\mathbf{min}_{\mathbb{R}^{A}}(\mathbf{max}_{\mathbb{R}^{A}}(q, p), q) = q$.

(46) $\mathbf{min}_{\mathbb{R}^{A}}(q, \mathbf{max}_{\mathbb{R}^{A}}(p, r)) = \mathbf{max}_{\mathbb{R}^{A}}(\mathbf{min}_{\mathbb{R}^{A}}(q, p), \mathbf{min}_{\mathbb{R}^{A}}(q, r))$.

(47) $\mathbb{R}^{A}_{\mathrm{L}}$ is a distributive lattice.

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Received May 22, 1990

Published January 2, 2004