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Quantales

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Summary. The concepts of Girard quantales (see [10] and [15]) and Blikle nets (see [3]) are introduced.

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The articles [12], [8], [13], [11], [14], [5], [4], [9], [16], [1], [2], [7], and [6] provide the notation and terminology for this paper.

Let X be a set and let Y be a subset of 2^X. Then $\bigcup Y$ is a subset of X.

In this article we present several logical schemes. The scheme *DenestFraenkel* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(a); a$ ranges over elements of $\mathcal{B} : a \in \{\mathcal{G}(b); b$ ranges over elements of $\mathcal{A} : \mathcal{P}[b]\}\} = \{\mathcal{F}(\mathcal{G}(a)); a$ ranges over elements of $\mathcal{A} : \mathcal{P}[a]\}$

for all values of the parameters.

The scheme *EmptyFraenkel* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(a); a$ ranges over elements of $\mathcal{A} : \mathcal{P}[a]\} = \emptyset$

provided the following requirement is met:

- It is not true that there exists an element a of $\mathcal{A}$ such that $\mathcal{P}[a]$.

We now state two propositions:

(1) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a_1, b_1 be elements of L_1, a_2, b_2 be elements of L_2, and X be a set. Suppose $a_1 = a_2$ and $b_1 = b_2$. Then $a_1 \sqcup b_1 = a_2 \sqcup b_2$ and $a_1 \sqcap b_1 = a_2 \sqcap b_2$ and $a_1 \sqsubseteq b_1$ iff $a_2 \sqsubseteq b_2$.

(2) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a be an element of L_1, b be an element of L_2, and X be a set such that $a = b$. Then

(i) $a \sqsubseteq X$ iff $b \sqsubseteq X$, and

(ii) $a \sqsupseteq X$ iff $b \sqsupseteq X$.

Let L be a 1-sorted structure. A unary operation on L is a map from L into L.

Let L be a non empty lattice structure and let X be a subset of L. We say that X is directed if and only if:

(Def. 1) For every finite subset Y of X there exists an element x of L such that $\bigsqcup_L Y \sqsubseteq x$ and $x \in X$.

One can prove the following proposition

(3) For every non empty lattice structure L and for every subset X of L such that X is directed holds X is non empty.

We introduce quantale structures which are extensions of lattice structure and groupoid and are systems

⟨ a carrier, a join operation, a meet operation, a multiplication ⟩,

where the carrier is a set and the join operation, the meet operation, and the multiplication are binary operations on the carrier.

Let us note that there exists a quantale structure which is non empty.

We consider quasinet structures as extensions of quantale structure and multiplicative loop structure as systems

⟨ a carrier, a join operation, a meet operation, a multiplication, a unity ⟩,

where the carrier is a set, the join operation, the meet operation, and the multiplication are binary operations on the carrier, and the unity is an element of the carrier.

Let us note that there exists a quasinet structure which is non empty.

Let I_1 be a non empty groupoid. We say that I_1 has left-zero if and only if:

(Def. 2) There exists an element a of I_1 such that for every element b of I_1 holds $a \cdot b = a$.

We say that I_1 has right-zero if and only if:

(Def. 3) There exists an element b of I_1 such that for every element a of I_1 holds $a \cdot b = b$.

Let I_1 be a non empty groupoid. We say that I_1 has zero if and only if:

(Def. 4) I_1 has left-zero and right-zero.

Let us observe that every non empty groupoid which has zero has also left-zero and right-zero and every non empty groupoid which has left-zero and right-zero has also zero.

Let us observe that there exists a non empty groupoid which has zero.

Let I_1 be a non empty quantale structure. We say that I_1 is right distributive if and only if:

(Def. 5) For every element a of I_1 and for every set X holds $a \otimes \bigsqcup_{I_1} X = \bigsqcup_{I_1} \{a \otimes b; b$ ranges over elements of I_1: $b \in X\}$.

We say that I_1 is left distributive if and only if:

(Def. 6) For every element a of I_1 and for every set X holds $\bigsqcup_{I_1} X \otimes a = \bigsqcup_{I_1} \{b \otimes a; b$ ranges over elements of I_1: $b \in X\}$.

We say that I_1 is $\otimes$-additive if and only if:

(Def. 7) For all elements a, b, c of I_1 holds $(a \sqcup b) \otimes c = a \otimes c \sqcup b \otimes c$ and $c \otimes (a \sqcup b) = c \otimes a \sqcup c \otimes b$.

We say that I_1 is $\otimes$-continuous if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let X_1, X_2 be subsets of I_1. Suppose X_1 is directed and X_2 is directed. Then $\bigsqcup X_1 \otimes \bigsqcup X_2 = \bigsqcup_{I_1} \{a \otimes b; a$ ranges over elements of I_1, b ranges over elements of I_1: $a \in X_1 \wedge b \in X_2\}$.

We now state the proposition

(4) Let Q be a non empty quantale structure. Suppose the lattice structure of $Q =$ the lattice of subsets of $\emptyset$. Then Q is associative, commutative, unital, complete, right distributive, left distributive, and lattice-like and has zero.

Let A be a non empty set and let b_1, b_2, b_3 be binary operations on A. Note that $\langle A, b_1, b_2, b_3 \rangle$ is non empty.

One can check that there exists a non empty quantale structure which is associative, commutative, unital, left distributive, right distributive, complete, and lattice-like and has zero.

The scheme *LUBFraenkelDistr* deals with a complete lattice-like non empty quantale structure $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and sets $\mathcal{B}$, $\mathcal{C}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(a,b); b$ ranges over elements of $\mathcal{A} : b \in \mathcal{C}\}; a$ ranges over elements of $\mathcal{A} : a \in \mathcal{B}\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(a,b); a$ ranges over elements of $\mathcal{A}, b$ ranges over elements of $\mathcal{A} : a \in \mathcal{B} \land b \in \mathcal{C}\}$

for all values of the parameters.

In the sequel Q denotes a left distributive right distributive complete lattice-like non empty quantale structure and a, b, c denote elements of Q.

Next we state two propositions:

(5) For every Q and for all sets X, Y holds $\bigsqcup_Q X \otimes \bigsqcup_Q Y = \bigsqcup_Q\{a \otimes b : a \in X \land b \in Y\}$.

(6) $(a \sqcup b) \otimes c = a \otimes c \sqcup b \otimes c$ and $c \otimes (a \sqcup b) = c \otimes a \sqcup c \otimes b$.

Let A be a non empty set, let b_1, b_2, b_3 be binary operations on A, and let e be an element of A. Observe that $\langle A, b_1, b_2, b_3, e\rangle$ is non empty.

One can verify that there exists a non empty quasinet structure which is complete and lattice-like.

Let us mention that every complete lattice-like non empty quasinet structure which is left distributive and right distributive is also $\otimes$-continuous and $\otimes$-additive.

Let us observe that there exists a non empty quasinet structure which is associative, commutative, well unital, left distributive, right distributive, complete, and lattice-like and has zero and left-zero.

A quantale is an associative left distributive right distributive complete lattice-like non empty quantale structure. A quasinet is a well unital associative $\otimes$-continuous $\otimes$-additive complete lattice-like non empty quasinet structure with left-zero.

A Blikle net is a non empty quasinet with zero.

The following proposition is true

(7) For every well unital non empty quasinet structure Q such that Q is a quantale holds Q is a Blikle net.

We use the following convention: Q denotes a quantale and a, b, c, d, D denote elements of Q.

Next we state two propositions:

(8) If $a \sqsubseteq b$, then $a \otimes c \sqsubseteq b \otimes c$ and $c \otimes a \sqsubseteq c \otimes b$.

(9) If $a \sqsubseteq b$ and $c \sqsubseteq d$, then $a \otimes c \sqsubseteq b \otimes d$.

Let f be a function. We say that f is idempotent if and only if:

(Def. 9) $f \cdot f = f$.

Let L be a non empty lattice structure and let I_1 be a unary operation on L. We say that I_1 is inflationary if and only if:

(Def. 10) For every element p of L holds $p \sqsubseteq I_1(p)$.

We say that I_1 is deflationary if and only if:

(Def. 11) For every element p of L holds $I_1(p) \sqsubseteq p$.

We say that I_1 is monotone if and only if:

(Def. 12) For all elements p, q of L such that $p \sqsubseteq q$ holds $I_1(p) \sqsubseteq I_1(q)$.

We say that I_1 is $\bigsqcup$-distributive if and only if:

(Def. 13) For every subset X of L holds $I_1(\bigsqcup X) \sqsubseteq \bigsqcup_L\{I_1(a); a$ ranges over elements of L: $a \in X\}$.

Let L be a lattice. One can check that there exists a unary operation on L which is inflationary, deflationary, and monotone.

Next we state the proposition

(10) Let L be a complete lattice and j be a unary operation on L. Suppose j is monotone. Then j is $\bigsqcup$-distributive if and only if for every subset X of L holds $j(\bigsqcup X) = \bigsqcup_L \{j(a); a$ ranges over elements of L: $a \in X\}$.

Let Q be a non empty quantale structure and let I_1 be a unary operation on Q. We say that I_1 is $\otimes$-monotone if and only if:

(Def. 14) For all elements a, b of Q holds $I_1(a) \otimes I_1(b) \sqsubseteq I_1(a \otimes b)$.

Let Q be a non empty quantale structure and let a, b be elements of Q. The functor $a \to_r b$ yielding an element of Q is defined as follows:

(Def. 15) $a \to_r b = \bigsqcup_Q \{c; c$ ranges over elements of Q: $c \otimes a \sqsubseteq b\}$.

The functor $a \to_l b$ yielding an element of Q is defined as follows:

(Def. 16) $a \to_l b = \bigsqcup_Q \{c; c$ ranges over elements of Q: $a \otimes c \sqsubseteq b\}$.

One can prove the following propositions:

(11) $a \otimes b \sqsubseteq c$ iff $b \sqsubseteq a \to_l c$.

(12) $a \otimes b \sqsubseteq c$ iff $a \sqsubseteq b \to_r c$.

(13) For every quantale Q and for all elements s, a, b of Q such that $a \sqsubseteq b$ holds $b \to_r s \sqsubseteq a \to_r s$ and $b \to_l s \sqsubseteq a \to_l s$.

(14) Let Q be a quantale, s be an element of Q, and j be a unary operation on Q. If for every element a of Q holds $j(a) = (a \to_r s) \to_r s$, then j is monotone.

Let Q be a non empty quantale structure and let I_1 be an element of Q. We say that I_1 is dualizing if and only if:

(Def. 17) For every element a of Q holds $(a \to_r I_1) \to_l I_1 = a$ and $(a \to_l I_1) \to_r I_1 = a$.

We say that I_1 is cyclic if and only if:

(Def. 18) For every element a of Q holds $a \to_r I_1 = a \to_l I_1$.

Next we state several propositions:

(15) c is cyclic iff for all a, b such that $a \otimes b \sqsubseteq c$ holds $b \otimes a \sqsubseteq c$.

(16) For every quantale Q and for all elements s, a of Q such that s is cyclic holds $a \sqsubseteq (a \to_r s) \to_r s$ and $a \sqsubseteq (a \to_l s) \to_l s$.

(17) For every quantale Q and for all elements s, a of Q such that s is cyclic holds $a \to_r s = ((a \to_r s) \to_r s) \to_r s$ and $a \to_l s = ((a \to_l s) \to_l s) \to_l s$.

(18) For every quantale Q and for all elements s, a, b of Q such that s is cyclic holds $((a \to_r s) \to_r s) \otimes ((b \to_r s) \to_r s) \sqsubseteq (a \otimes b \to_r s) \to_r s$.

(19) If D is dualizing, then Q is unital and $\mathbf{1}_{\text{the multiplication of } Q} = D \to_r D$ and $\mathbf{1}_{\text{the multiplication of } Q} = D \to_l D$.

(20) If a is dualizing, then $b \to_r c = b \otimes (c \to_l a) \to_r a$ and $b \to_l c = (c \to_r a) \otimes b \to_l a$.

We introduce Girard quantale structures which are extensions of quasinet structure and are systems

$\langle$ a carrier, a join operation, a meet operation, a multiplication, a unity, an absurd $\rangle$,

where the carrier is a set, the join operation, the meet operation, and the multiplication are binary operations on the carrier, and the unity and the absurd are elements of the carrier.

Let us note that there exists a Girard quantale structure which is non empty.

Let I_1 be a non empty Girard quantale structure. We say that I_1 is cyclic if and only if:

(Def. 19) The absurd of I_1 is cyclic.

We say that I_1 is dualized if and only if:

(Def. 20) The absurd of I_1 is dualizing.

We now state the proposition

(21) Let Q be a non empty Girard quantale structure. Suppose the lattice structure of $Q =$ the lattice of subsets of $\emptyset$. Then Q is cyclic and dualized.

Let A be a non empty set, let b_1, b_2, b_3 be binary operations on A, and let e_1, e_2 be elements of A. Observe that $\langle A, b_1, b_2, b_3, e_1, e_2 \rangle$ is non empty.

Let us note that there exists a non empty Girard quantale structure which is associative, commutative, well unital, left distributive, right distributive, complete, lattice-like, cyclic, dualized, and strict.

A Girard quantale is an associative well unital left distributive right distributive complete lattice-like cyclic dualized non empty Girard quantale structure.

Let G be a Girard quantale structure. The functor $\perp_G$ yields an element of G and is defined by:

(Def. 21) $\perp_G =$ the absurd of G.

Let G be a non empty Girard quantale structure. The functor $\top_G$ yields an element of G and is defined by:

(Def. 22) $\top_G = \perp_G \rightarrow_r \perp_G$.

Let a be an element of G. The functor $\perp_a$ yields an element of G and is defined as follows:

(Def. 23) $\perp_a = a \rightarrow_r \perp_G$.

Let G be a non empty Girard quantale structure. The functor $\mathrm{Negation}(G)$ yields a unary operation on G and is defined by:

(Def. 24) For every element a of G holds $(\mathrm{Negation}(G))(a) = \perp_a$.

Let G be a non empty Girard quantale structure and let u be a unary operation on G. The functor $\perp_u$ yields a unary operation on G and is defined as follows:

(Def. 25) $\perp_u = \mathrm{Negation}(G) \cdot u$.

Let G be a non empty Girard quantale structure and let o be a binary operation on G. The functor $\perp_o$ yields a binary operation on G and is defined by:

(Def. 26) $\perp_o = \mathrm{Negation}(G) \cdot o$.

We use the following convention: Q denotes a Girard quantale, a, a_1, a_2, b, b_1, b_2, c denote elements of Q, and X denotes a set.

One can prove the following propositions:

(22) $\perp_{\perp_a} = a$.

(23) If $a \sqsubseteq b$, then $\perp_b \sqsubseteq \perp_a$.

(24) $\perp_{\bigsqcup_Q X} = \bigsqcap_Q \{\perp_a : a \in X\}$.

(25) $\perp_{\bigsqcap_Q X} = \bigsqcup_Q \{\perp_a : a \in X\}$.

(26) $\perp_{a \sqcup b} = \perp_a \sqcap \perp_b$ and $\perp_{a \sqcap b} = \perp_a \sqcup \perp_b$.

Let us consider Q, a, b. The functor $a \,\mathrm{delta}\, b$ yields an element of Q and is defined as follows:

(Def. 27) $a \,\mathrm{delta}\, b = \perp_{\perp_a \otimes \perp_b}$.

One can prove the following propositions:

(27) $a \otimes \bigsqcup_Q X = \bigsqcup_Q \{a \otimes b : b \in X\}$ and $a \,\mathrm{delta} \bigsqcap_Q X = \bigsqcap_Q \{a \,\mathrm{delta}\, c : c \in X\}$.

(28) $\bigsqcup_Q X \otimes a = \bigsqcup_Q \{b \otimes a : b \in X\}$ and $\bigsqcap_Q X \,\mathrm{delta}\, a = \bigsqcap_Q \{c \,\mathrm{delta}\, a : c \in X\}$.

(29) $a \,\mathrm{delta}\, b \sqcap c = (a \,\mathrm{delta}\, b) \sqcap (a \,\mathrm{delta}\, c)$ and $b \sqcap c \,\mathrm{delta}\, a = (b \,\mathrm{delta}\, a) \sqcap (c \,\mathrm{delta}\, a)$.

(30) If $a_1 \sqsubseteq b_1$ and $a_2 \sqsubseteq b_2$, then $a_1 \,\mathrm{delta}\, a_2 \sqsubseteq b_1 \,\mathrm{delta}\, b_2$.

(31) $a \,\mathrm{delta}\, b \,\mathrm{delta}\, c = a \,\mathrm{delta}\, b \,\mathrm{delta}\, c$.

(32) $a \otimes \top_Q = a$ and $\top_Q \otimes a = a$.

(33) $a \,\mathrm{delta} \perp_Q = a$ and $\perp_Q \mathrm{delta}\, a = a$.

(34) Let Q be a quantale and j be a unary operation on Q. Suppose j is monotone, idempotent, and $\bigsqcup$-distributive. Then there exists a complete lattice L such that the carrier of $L = \mathrm{rng}\, j$ and for every subset X of L holds $\bigsqcup X = j(\bigsqcup_Q X)$.

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