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The Fundamental Logic Structure in Quantum Mechanics

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Summary. In this article we present the logical structure given by four axioms of Mackey [4] in the set of propositions of Quantum Mechanics. The equivalence relation (PropRel(Q)) in the set of propositions (Prop Q) for given Quantum Mechanics Q is considered. The main text for this article is [6] where the structure of quotient space and the properties of equivalence relations, classes and partitions are studied.

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The articles [8], [3], [12], [10], [13], [14], [15], [1], [2], [11], [7], [5], [9], and [6] provide the notation and terminology for this paper.

In this paper X_1, x are sets, X is a non empty set, and A is an event of the Borel sets.

Let us consider X and let S be a σ-field of subsets of X. The functor probabilities S yielding a set is defined by:

(Def. 1) $x \in$ probabilities S iff x is a probability on S.

Let us consider X and let S be a σ-field of subsets of X. One can check that probabilities S is non empty.

We consider quantum mechanics structures as systems
⟨ observables, control states, a probability ⟩,
where the observables and the control states constitute non empty sets and the probability is a function from [: the observables, the control states :] into probabilities (the Borel sets).

In the sequel Q denotes a quantum mechanics structure.

Let us consider Q. The functor Obs Q yielding a set is defined by:

(Def. 2) Obs $Q =$ the observables of Q.

The functor Sts Q yields a set and is defined as follows:

(Def. 3) Sts $Q =$ the control states of Q.

Let us consider Q. One can verify that Obs Q is non empty and Sts Q is non empty.

In the sequel A_1 is an element of Obs Q, s is an element of Sts Q, and E is an event of the Borel sets.

Let us consider Q, A_1, s. The functor $\mathrm{Meas}(A_1, s)$ yields a probability on the Borel sets and is defined by:

(Def. 4) $\mathrm{Meas}(A_1, s) =$ (the probability of Q)$(\langle A_1, s\rangle)$.

Let I_1 be a quantum mechanics structure. We say that I_1 is quantum mechanics-like if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For all elements A_1, A_2 of $\mathrm{Obs}\, I_1$ such that for every element s of $\mathrm{Sts}\, I_1$ holds $\mathrm{Meas}(A_1, s) = \mathrm{Meas}(A_2, s)$ holds $A_1 = A_2$,

(ii) for all elements s_1, s_2 of $\mathrm{Sts}\, I_1$ such that for every element A of $\mathrm{Obs}\, I_1$ holds $\mathrm{Meas}(A, s_1) = \mathrm{Meas}(A, s_2)$ holds $s_1 = s_2$, and

(iii) for all elements s_1, s_2 of $\mathrm{Sts}\, I_1$ and for every real number t such that $0 \leq t$ and $t \leq 1$ there exists an element s of $\mathrm{Sts}\, I_1$ such that for every element A of $\mathrm{Obs}\, I_1$ and for every E holds $\mathrm{Meas}(A, s)(E) = t \cdot \mathrm{Meas}(A, s_1)(E) + (1 - t) \cdot \mathrm{Meas}(A, s_2)(E)$.

Let us note that there exists a quantum mechanics structure which is strict and quantum mechanics-like.

A quantum mechanics is a quantum mechanics-like quantum mechanics structure.

In the sequel Q denotes a quantum mechanics and s denotes an element of $\mathrm{Sts}\, Q$.

Let X be a set. We consider POI structures over X as systems

$\langle$ an ordering, an involution $\rangle$,

where the ordering is a binary relation on X and the involution is a function from X into X.

In the sequel x_1 is an element of X_1 and I_2 is a function from X_1 into X_1.

Let us consider X_1, I_2. We say that I_2 is an involution in X_1 if and only if:

(Def. 6) $I_2(I_2(x_1)) = x_1$.

Let us consider X_1 and let W be a POI structure over X_1. We say that W is a quantum logic on X_1 if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a binary relation O_1 on X_1 and there exists a function I_2 from X_1 into X_1 such that

(i) $W = \langle O_1, I_2 \rangle$,

(ii) O_1 partially orders X_1,

(iii) I_2 is an involution in X_1, and

(iv) for all elements x, y of X_1 such that $\langle x, y \rangle \in O_1$ holds $\langle I_2(y), I_2(x) \rangle \in O_1$.

Let us consider Q. The functor $\mathrm{Prop}\, Q$ yields a set and is defined as follows:

(Def. 8) $\mathrm{Prop}\, Q = [\!:\mathrm{Obs}\, Q, \text{the Borel sets}:\!]$.

Let us consider Q. Observe that $\mathrm{Prop}\, Q$ is non empty.

In the sequel p, q, r, p_1, q_1 denote elements of $\mathrm{Prop}\, Q$.

Let us consider Q, p. Then $p_{\mathbf{1}}$ is an element of $\mathrm{Obs}\, Q$. Then $p_{\mathbf{2}}$ is an event of the Borel sets.

We now state two propositions:

(14)[1] $p = \langle p_{\mathbf{1}}, p_{\mathbf{2}} \rangle$.

(16)[2] For every E such that $E = (p_{\mathbf{2}})^{\mathrm{c}}$ holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) = 1 - \mathrm{Meas}(p_{\mathbf{1}}, s)(E)$.

Let us consider Q, p. The functor $\neg p$ yielding an element of $\mathrm{Prop}\, Q$ is defined by:

(Def. 9) $\neg p = \langle p_{\mathbf{1}}, (p_{\mathbf{2}})^{\mathrm{c}} \rangle$.

Let us consider Q, p, q. The predicate $p \vdash q$ is defined by:

(Def. 10) For every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) \leq \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

Let us consider Q, p, q. The predicate $p \equiv q$ is defined by:

(Def. 11) $p \vdash q$ and $q \vdash p$.

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[1] The propositions (1)–(13) have been removed.

[2] The proposition (15) has been removed.

One can prove the following propositions:

(20)[3] $p \equiv q$ iff for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) = \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

(21) $p \vdash p$.

(22) If $p \vdash q$ and $q \vdash r$, then $p \vdash r$.

(23) $p \equiv p$.

(24) If $p \equiv q$, then $q \equiv p$.

(25) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(26) $(\neg p)_{\mathbf{1}} = p_{\mathbf{1}}$ and $(\neg p)_{\mathbf{2}} = (p_{\mathbf{2}})^{\mathrm{c}}$.

(27) $\neg\neg p = p$.

(28) If $p \vdash q$, then $\neg q \vdash \neg p$.

Let us consider Q. The functor $\mathrm{PropRel}\,Q$ yielding an equivalence relation of $\mathrm{Prop}\,Q$ is defined by:

(Def. 12) $\langle p, q \rangle \in \mathrm{PropRel}\,Q$ iff $p \equiv q$.

In the sequel B, C are subsets of $\mathrm{Prop}\,Q$.
Next we state the proposition

(30)[4] Let given B, C. Suppose $B \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and $C \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$. Let a, b, c, d be elements of $\mathrm{Prop}\,Q$. If $a \in B$ and $b \in B$ and $c \in C$ and $d \in C$ and $a \vdash c$, then $b \vdash d$.

Let us consider Q. The functor $\mathrm{OrdRel}\,Q$ yields a binary relation on $\mathrm{Classes}\,\mathrm{PropRel}\,Q$ and is defined by:

(Def. 13) $\langle B, C \rangle \in \mathrm{OrdRel}\,Q$ iff $B \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and $C \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and for all p, q such that $p \in B$ and $q \in C$ holds $p \vdash q$.

The following propositions are true:

(32)[5] $p \vdash q$ iff $\langle [p]_{\mathrm{PropRel}\,Q}, [q]_{\mathrm{PropRel}\,Q} \rangle \in \mathrm{OrdRel}\,Q$.

(33) For all B, C such that $B \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and $C \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and for all p_1, q_1 such that $p_1 \in B$ and $q_1 \in B$ and $\neg p_1 \in C$ holds $\neg q_1 \in C$.

(34) For all B, C such that $B \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and $C \in \mathrm{Classes}\,\mathrm{PropRel}\,Q$ and for all p, q such that $\neg p \in C$ and $\neg q \in C$ and $p \in B$ holds $q \in B$.

Let us consider Q. The functor $\mathrm{InvRel}\,Q$ yielding a function from $\mathrm{Classes}\,\mathrm{PropRel}\,Q$ into $\mathrm{Classes}\,\mathrm{PropRel}\,Q$ is defined by:

(Def. 14) $(\mathrm{InvRel}\,Q)([p]_{\mathrm{PropRel}\,Q}) = [\neg p]_{\mathrm{PropRel}\,Q}$.

We now state the proposition

(36)[6] For every Q holds $\langle \mathrm{OrdRel}\,Q, \mathrm{InvRel}\,Q \rangle$ is a quantum logic on $\mathrm{Classes}\,\mathrm{PropRel}\,Q$.

[3] The propositions (17)–(19) have been removed.
[4] The proposition (29) has been removed.
[5] The proposition (31) has been removed.
[6] The proposition (35) has been removed.

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