JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Connectives and Subformulae of the First Order Language

Grzegorz Bancerek
Warsaw University
Białystok

**Summary.** In the article the development of the first order language defined in [4] is continued. The following connectives are introduced: implication ($\Rightarrow$), disjunction ($\vee$), and equivalence ($\Leftrightarrow$). We introduce also the existential quantifier ($\exists$) and FALSUM. Some theorems on disjunctive, conditional, biconditional and existential formulae are proved and their selector functors are introduced. The second part of the article deals with notions of subformula, proper subformula and immediate constituent of a QC-formula.

MML Identifier: `QC_LANG2`.

WWW: `http://mizar.org/JFM/Vol1/qc_lang2.html`

The articles [6], [5], [8], [9], [3], [1], [2], [7], and [4] provide the notation and terminology for this paper.

We use the following convention: $x$, $y$, $z$ denote bound variables and $p$, $q$, $p_1$, $p_2$, $q_1$ denote elements of WFF.

The following propositions are true:

(1) If $\neg p = \neg q$, then $p = q$.

(2) $\mathrm{Arg}(\neg p) = p$.

(3) If $p \wedge q = p_1 \wedge q_1$, then $p = p_1$ and $q = q_1$.

(4) If $p$ is conjunctive, then $p = \mathrm{LeftArg}(p) \wedge \mathrm{RightArg}(p)$.

(5) $\mathrm{LeftArg}(p \wedge q) = p$ and $\mathrm{RightArg}(p \wedge q) = q$.

(6) If $\forall_x p = \forall_y q$, then $x = y$ and $p = q$.

(7) If $p$ is universal, then $p = \forall_{\mathrm{Bound}(p)} \mathrm{Scope}(p)$.

(8) $\mathrm{Bound}(\forall_x p) = x$ and $\mathrm{Scope}(\forall_x p) = p$.

The formula FALSUM is defined as follows:

(Def. 1) $\mathrm{FALSUM} = \neg \mathrm{VERUM}$.

Let $p$, $q$ be elements of WFF. The functor $p \Rightarrow q$ yields a formula and is defined by:

(Def. 2) $p \Rightarrow q = \neg(p \wedge \neg q)$.

The functor $p \vee q$ yielding a formula is defined as follows:

(Def. 3) $p \vee q = \neg(\neg p \wedge \neg q)$.

Let $p$, $q$ be elements of WFF. The functor $p \Leftrightarrow q$ yielding a formula is defined as follows:

(Def. 4) $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Let $x$ be a bound variable and let $p$ be an element of WFF. The functor $\exists_x p$ yields a formula and is defined as follows:

(Def. 5) $\exists_x p = \neg\forall_x \neg p$.

Next we state several propositions:

(13)[1] FALSUM is negative and $\mathrm{Arg}(\mathrm{FALSUM}) = \mathrm{VERUM}$.

(14) $p \vee q = \neg p \Rightarrow q$.

(16)[2] If $p \vee q = p_1 \vee q_1$, then $p = p_1$ and $q = q_1$.

(17) If $p \Rightarrow q = p_1 \Rightarrow q_1$, then $p = p_1$ and $q = q_1$.

(18) If $p \Leftrightarrow q = p_1 \Leftrightarrow q_1$, then $p = p_1$ and $q = q_1$.

(19) If $\exists_x p = \exists_y q$, then $x = y$ and $p = q$.

Let $x$, $y$ be bound variables and let $p$ be an element of WFF. The functor $\forall_{x,y} p$ yielding a formula is defined by:

(Def. 6) $\forall_{x,y} p = \forall_x \forall_y p$.

The functor $\exists_{x,y} p$ yields a formula and is defined by:

(Def. 7) $\exists_{x,y} p = \exists_x \exists_y p$.

Next we state several propositions:

(20) $\forall_{x,y} p = \forall_x \forall_y p$ and $\exists_{x,y} p = \exists_x \exists_y p$.

(21) For all bound variables $x_1$, $x_2$, $y_1$, $y_2$ such that $\forall_{x_1,y_1} p_1 = \forall_{x_2,y_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(22) If $\forall_{x,y} p = \forall_z q$, then $x = z$ and $\forall_y p = q$.

(23) For all bound variables $x_1$, $x_2$, $y_1$, $y_2$ such that $\exists_{x_1,y_1} p_1 = \exists_{x_2,y_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(24) If $\exists_{x,y} p = \exists_z q$, then $x = z$ and $\exists_y p = q$.

(25) $\forall_{x,y} p$ is universal and $\mathrm{Bound}(\forall_{x,y} p) = x$ and $\mathrm{Scope}(\forall_{x,y} p) = \forall_y p$.

Let $x$, $y$, $z$ be bound variables and let $p$ be an element of WFF. The functor $\forall_{x,y,z} p$ yielding a formula is defined by:

(Def. 8) $\forall_{x,y,z} p = \forall_x \forall_{y,z} p$.

The functor $\exists_{x,y,z} p$ yielding a formula is defined by:

(Def. 9) $\exists_{x,y,z} p = \exists_x \exists_{y,z} p$.

One can prove the following propositions:

(26) $\forall_{x,y,z} p = \forall_x \forall_{y,z} p$ and $\exists_{x,y,z} p = \exists_x \exists_{y,z} p$.

---

[1] The propositions (9)–(12) have been removed.
[2] The proposition (15) has been removed.

(27) For all bound variables $x_1$, $x_2$, $y_1$, $y_2$, $z_1$, $z_2$ such that $\forall_{x_1,y_1,z_1} p_1 = \forall_{x_2,y_2,z_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

In the sequel $s$, $t$ are bound variables.
One can prove the following propositions:

(28) If $\forall_{x,y,z} p = \forall_t q$, then $x = t$ and $\forall_{y,z} p = q$.

(29) If $\forall_{x,y,z} p = \forall_{t,s} q$, then $x = t$ and $y = s$ and $\forall_z p = q$.

(30) For all bound variables $x_1$, $x_2$, $y_1$, $y_2$, $z_1$, $z_2$ such that $\exists_{x_1,y_1,z_1} p_1 = \exists_{x_2,y_2,z_2} p_2$ holds $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(31) If $\exists_{x,y,z} p = \exists_t q$, then $x = t$ and $\exists_{y,z} p = q$.

(32) If $\exists_{x,y,z} p = \exists_{t,s} q$, then $x = t$ and $y = s$ and $\exists_z p = q$.

(33) $\forall_{x,y,z} p$ is universal and $\mathrm{Bound}(\forall_{x,y,z} p) = x$ and $\mathrm{Scope}(\forall_{x,y,z} p) = \forall_{y,z} p$.

Let $H$ be an element of WFF. We say that $H$ is disjunctive if and only if:

(Def. 10) There exist elements $p$, $q$ of WFF such that $H = p \vee q$.

We say that $H$ is conditional if and only if:

(Def. 11) There exist elements $p$, $q$ of WFF such that $H = p \Rightarrow q$.

We say that $H$ is biconditional if and only if:

(Def. 12) There exist elements $p$, $q$ of WFF such that $H = p \Leftrightarrow q$.

We say that $H$ is existential if and only if:

(Def. 13) There exists a bound variable $x$ and there exists an element $p$ of WFF such that $H = \exists_x p$.

One can prove the following proposition

(38)[3] $\exists_{x,y} p$ is existential and $\exists_{x,y,z} p$ is existential.

Let $H$ be an element of WFF. The functor $\mathrm{LeftDisj}(H)$ yields a formula and is defined by:

(Def. 14) $\mathrm{LeftDisj}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

The functor $\mathrm{RightDisj}(H)$ yields a formula and is defined by:

(Def. 15) $\mathrm{RightDisj}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

We introduce $\mathrm{Consequent}(H)$ as a synonym of $\mathrm{RightDisj}(H)$. The functor $\mathrm{Antecedent}(H)$ yields a formula and is defined by:

(Def. 16) $\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

Let $H$ be an element of WFF. The functor $\mathrm{LeftSide}(H)$ yields a formula and is defined as follows:

(Def. 18)[4] $\mathrm{LeftSide}(H) = \mathrm{Antecedent}(\mathrm{LeftArg}(H))$.

The functor $\mathrm{RightSide}(H)$ yielding a formula is defined by:

(Def. 19) $\mathrm{RightSide}(H) = \mathrm{Consequent}(\mathrm{LeftArg}(H))$.

In the sequel $F$, $G$, $H$ denote elements of WFF.
Next we state a number of propositions:

[3] The propositions (34)–(37) have been removed.
[4] The definition (Def. 17) has been removed.

(45)[5] LeftDisj$(F \vee G) = F$ and RightDisj$(F \vee G) = G$ and Arg$(F \vee G) = \neg F \wedge \neg G$.

(46) Antecedent$(F \Rightarrow G) = F$ and Consequent$(F \Rightarrow G) = G$ and Arg$(F \Rightarrow G) = F \wedge \neg G$.

(47) LeftSide$(F \Leftrightarrow G) = F$ and RightSide$(F \Leftrightarrow G) = G$ and LeftArg$(F \Leftrightarrow G) = F \Rightarrow G$ and RightArg$(F \Leftrightarrow G) = G \Rightarrow F$.

(48) Arg$(\exists_x H) = \forall_x \neg H$.

(49) Suppose $H$ is disjunctive. Then $H$ is conditional and negative and Arg$(H)$ is conjunctive and LeftArg(Arg$(H)$) is negative and RightArg(Arg$(H)$) is negative.

(50) If $H$ is conditional, then $H$ is negative and Arg$(H)$ is conjunctive and RightArg(Arg$(H)$) is negative.

(51) If $H$ is biconditional, then $H$ is conjunctive and LeftArg$(H)$ is conditional and RightArg$(H)$ is conditional.

(52) If $H$ is existential, then $H$ is negative and Arg$(H)$ is universal and Scope(Arg$(H)$) is negative.

(53) If $H$ is disjunctive, then $H =$ LeftDisj$(H) \vee$ RightDisj$(H)$.

(54) If $H$ is conditional, then $H =$ Antecedent$(H) \Rightarrow$ Consequent$(H)$.

(55) If $H$ is biconditional, then $H =$ LeftSide$(H) \Leftrightarrow$ RightSide$(H)$.

(56) If $H$ is existential, then $H = \exists_{\mathrm{Bound}(\mathrm{Arg}(H))} \mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(H)))$.

Let $G$, $H$ be elements of WFF. We say that $G$ is an immediate constituent of $H$ if and only if:

(Def. 20) $H = \neg G$ or there exists an element $F$ of WFF such that $H = G \wedge F$ or $H = F \wedge G$ or there exists a bound variable $x$ such that $H = \forall_x G$.

For simplicity, we adopt the following convention: $x$ is a bound variable, $k$, $n$ are natural numbers, $P$ is a $k$-ary predicate symbol, and $V$ is a list of variables of the length $k$.

One can prove the following propositions:

(58)[6] $H$ is not an immediate constituent of VERUM.

(59) $H$ is not an immediate constituent of $P[V]$.

(60) $F$ is an immediate constituent of $\neg H$ iff $F = H$.

(61) $H$ is an immediate constituent of FALSUM iff $H =$ VERUM.

(62) $F$ is an immediate constituent of $G \wedge H$ iff $F = G$ or $F = H$.

(63) $F$ is an immediate constituent of $\forall_x H$ iff $F = H$.

(64) If $H$ is atomic, then $F$ is not an immediate constituent of $H$.

(65) If $H$ is negative, then $F$ is an immediate constituent of $H$ iff $F =$ Arg$(H)$.

(66) If $H$ is conjunctive, then $F$ is an immediate constituent of $H$ iff $F =$ LeftArg$(H)$ or $F =$ RightArg$(H)$.

(67) If $H$ is universal, then $F$ is an immediate constituent of $H$ iff $F =$ Scope$(H)$.

In the sequel $L$ denotes a finite sequence.

Let us consider $G$, $H$. We say that $G$ is a subformula of $H$ if and only if the condition (Def. 21) is satisfied.

[5] The propositions (39)–(44) have been removed.

[6] The proposition (57) has been removed.

(Def. 21) There exist $n$, $L$ such that

(i) $1 \leq n$,

(ii) $\operatorname{len} L = n$,

(iii) $L(1) = G$,

(iv) $L(n) = H$, and

(v) for every $k$ such that $1 \leq k$ and $k < n$ there exist elements $G_1$, $H_1$ of WFF such that $L(k) = G_1$ and $L(k+1) = H_1$ and $G_1$ is an immediate constituent of $H_1$.

Let us note that the predicate $G$ is a subformula of $H$ is reflexive.

Let us consider $H$, $F$. We say that $H$ is a proper subformula of $F$ if and only if:

(Def. 22) $H$ is a subformula of $F$ and $H \neq F$.

Let us note that the predicate $H$ is a proper subformula of $F$ is irreflexive.

Next we state a number of propositions:

(71)[7] If $H$ is an immediate constituent of $F$, then $\operatorname{len}({}^{@}H) < \operatorname{len}({}^{@}F)$.

(72) If $H$ is an immediate constituent of $F$, then $H$ is a subformula of $F$.

(73) If $H$ is an immediate constituent of $F$, then $H$ is a proper subformula of $F$.

(74) If $H$ is a proper subformula of $F$, then $\operatorname{len}({}^{@}H) < \operatorname{len}({}^{@}F)$.

(75) If $H$ is a proper subformula of $F$, then there exists $G$ which is an immediate constituent of $F$.

(76) If $F$ is a proper subformula of $G$ and $G$ is a proper subformula of $H$, then $F$ is a proper subformula of $H$.

(77) If $F$ is a subformula of $G$ and $G$ is a subformula of $H$, then $F$ is a subformula of $H$.

(78) If $G$ is a subformula of $H$ and $H$ is a subformula of $G$, then $G = H$.

(79) $G$ is not a proper subformula of $H$ or $H$ is not a subformula of $G$.

(80) $G$ is not a proper subformula of $H$ or $H$ is not a proper subformula of $G$.

(81) $G$ is not a subformula of $H$ or $H$ is not an immediate constituent of $G$.

(82) $G$ is not a proper subformula of $H$ or $H$ is not an immediate constituent of $G$.

(83) Suppose that

(i) $F$ is a proper subformula of $G$ and $G$ is a subformula of $H$, or

(ii) $F$ is a subformula of $G$ and $G$ is a proper subformula of $H$, or

(iii) $F$ is a subformula of $G$ and $G$ is an immediate constituent of $H$, or

(iv) $F$ is an immediate constituent of $G$ and $G$ is a subformula of $H$, or

(v) $F$ is a proper subformula of $G$ and $G$ is an immediate constituent of $H$, or

(vi) $F$ is an immediate constituent of $G$ and $G$ is a proper subformula of $H$.

Then $F$ is a proper subformula of $H$.

(84) $F$ is not a proper subformula of VERUM.

(85) $F$ is not a proper subformula of $P[V]$.

(86) $F$ is a subformula of $H$ iff $F$ is a proper subformula of $\neg H$.

[7] The propositions (68)–(70) have been removed.

(87) If $\neg F$ is a subformula of $H$, then $F$ is a proper subformula of $H$.

(88) $F$ is a proper subformula of FALSUM iff $F$ is a subformula of VERUM.

(89) $F$ is a subformula of $G$ and a subformula of $H$ iff $F$ is a proper subformula of $G \wedge H$.

(90) If $F \wedge G$ is a subformula of $H$, then $F$ is a proper subformula of $H$ and $G$ is a proper subformula of $H$.

(91) $F$ is a subformula of $H$ iff $F$ is a proper subformula of $\forall_x H$.

(92) If $\forall_x H$ is a subformula of $F$, then $H$ is a proper subformula of $F$.

(93)(i) $F \wedge \neg G$ is a proper subformula of $F \Rightarrow G$,
(ii) $F$ is a proper subformula of $F \Rightarrow G$,
(iii) $\neg G$ is a proper subformula of $F \Rightarrow G$, and
(iv) $G$ is a proper subformula of $F \Rightarrow G$.

(94)(i) $\neg F \wedge \neg G$ is a proper subformula of $F \vee G$,
(ii) $\neg F$ is a proper subformula of $F \vee G$,
(iii) $\neg G$ is a proper subformula of $F \vee G$,
(iv) $F$ is a proper subformula of $F \vee G$, and
(v) $G$ is a proper subformula of $F \vee G$.

(95) If $H$ is atomic, then $F$ is not a proper subformula of $H$.

(96) If $H$ is negative, then $\mathrm{Arg}(H)$ is a proper subformula of $H$.

(97) If $H$ is conjunctive, then $\mathrm{LeftArg}(H)$ is a proper subformula of $H$ and $\mathrm{RightArg}(H)$ is a proper subformula of $H$.

(98) If $H$ is universal, then $\mathrm{Scope}(H)$ is a proper subformula of $H$.

(99) $H$ is a subformula of VERUM iff $H = \mathrm{VERUM}$.

(100) $H$ is a subformula of $P[V]$ iff $H = P[V]$.

(101) $H$ is a subformula of FALSUM iff $H = \mathrm{FALSUM}$ or $H = \mathrm{VERUM}$.

Let us consider $H$. The functor Subformulae$H$ yields a set and is defined as follows:

(Def. 23) For every set $a$ holds $a \in \mathrm{Subformulae}H$ iff there exists $F$ such that $F = a$ and $F$ is a subformula of $H$.

We now state a number of propositions:

(103)[8] If $G \in \mathrm{Subformulae}H$, then $G$ is a subformula of $H$.

(104) If $F$ is a subformula of $H$, then $\mathrm{Subformulae}F \subseteq \mathrm{Subformulae}H$.

(105) If $G \in \mathrm{Subformulae}H$, then $\mathrm{Subformulae}G \subseteq \mathrm{Subformulae}H$.

(107)[9] $\mathrm{Subformulae}\,\mathrm{VERUM} = \{\mathrm{VERUM}\}$.

(108) $\mathrm{Subformulae}(P[V]) = \{P[V]\}$.

(109) $\mathrm{Subformulae}\,\mathrm{FALSUM} = \{\mathrm{VERUM}, \mathrm{FALSUM}\}$.

(110) $\mathrm{Subformulae}\,\neg H = \mathrm{Subformulae}H \cup \{\neg H\}$.

[8] The proposition (102) has been removed.
[9] The proposition (106) has been removed.

(111) $\mathrm{Subformulae}(H \wedge F) = \mathrm{Subformulae}\,H \cup \mathrm{Subformulae}\,F \cup \{H \wedge F\}$.

(112) $\mathrm{Subformulae}\,\forall_x H = \mathrm{Subformulae}\,H \cup \{\forall_x H\}$.

(113) $\mathrm{Subformulae}(F \Rightarrow G) = \mathrm{Subformulae}\,F \cup \mathrm{Subformulae}\,G \cup \{\neg G, F \wedge \neg G, F \Rightarrow G\}$.

(114) $\mathrm{Subformulae}(F \vee G) = \mathrm{Subformulae}\,F \cup \mathrm{Subformulae}\,G \cup \{\neg G, \neg F, \neg F \wedge \neg G, F \vee G\}$.

(115) $\mathrm{Subformulae}(F \Leftrightarrow G) = \mathrm{Subformulae}\,F \cup \mathrm{Subformulae}\,G \cup \{\neg G, F \wedge \neg G, F \Rightarrow G, \neg F, G \wedge \neg F, G \Rightarrow F, F \Leftrightarrow G\}$.

(116) $H = \mathrm{VERUM}$ or $H$ is atomic iff $\mathrm{Subformulae}\,H = \{H\}$.

(117) If $H$ is negative, then $\mathrm{Subformulae}\,H = \mathrm{Subformulae}\,\mathrm{Arg}(H) \cup \{H\}$.

(118) If $H$ is conjunctive, then $\mathrm{Subformulae}\,H = \mathrm{Subformulae}\,\mathrm{LeftArg}(H) \cup \mathrm{Subformulae}\,\mathrm{RightArg}(H) \cup \{H\}$.

(119) If $H$ is universal, then $\mathrm{Subformulae}\,H = \mathrm{Subformulae}\,\mathrm{Scope}(H) \cup \{H\}$.

(120) Suppose $H$ is an immediate constituent of $G$, a proper subformula of $G$, and a subformula of $G$ and $G \in \mathrm{Subformulae}\,F$. Then $H \in \mathrm{Subformulae}\,F$.

## REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Piotr Rudnicki and Andrzej Trybulec. A first order language. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/qc_lang1.html`.

[5] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/enumset1.html`.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[7] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[9] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

*Received November 23, 1989*

*Published January 2, 2004*