JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

A First Order Language

Piotr Rudnicki
The University of Alberta

Andrzej Trybulec
Warsaw University
Białystok

Summary. In the paper a first order language is constructed. It includes the universal quantifier and the following propositional connectives: truth, negation, and conjunction. The variables are divided into three kinds: bound variables, fixed variables, and free variables. An infinite number of predicates for each arity is provided. Schemes of structural induction and schemes justifying definitions by structural induction have been proved. The concept of a closed formula (a formula without free occurrences of bound variables) is introduced.

MML Identifier: QC_LANG1.

WWW: http://mizar.org/JFM/Vol1/qc_lang1.html

The articles [7], [6], [5], [10], [9], [8], [1], [11], [3], [12], [4], and [2] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) For every non empty set D_1 and for every set D_2 and for every element k of D_1 holds $[: \{k\}, D_2 :] \subseteq [: D_1, D_2 :]$.

(2) For every non empty set D_1 and for every set D_2 and for all elements k_1, k_2, k_3 of D_1 holds $[: \{k_1, k_2, k_3\}, D_2 :] \subseteq [: D_1, D_2 :]$.

In the sequel k, l denote natural numbers.

The set Var is defined as follows:

(Def. 1) $\mathrm{Var} = [: \{4, 5, 6\}, \mathbb{N} :]$.

Let us observe that Var is non empty.

One can prove the following proposition

(4)[1] $\mathrm{Var} \subseteq [: \mathbb{N}, \mathbb{N} :]$.

A variable is an element of Var. The subset BoundVar of Var is defined as follows:

(Def. 2) $\mathrm{BoundVar} = [: \{4\}, \mathbb{N} :]$.

The subset FixedVar of Var is defined by:

(Def. 3) $\mathrm{FixedVar} = [: \{5\}, \mathbb{N} :]$.

The subset FreeVar of Var is defined by:

(Def. 4) $\mathrm{FreeVar} = [: \{6\}, \mathbb{N} :]$.

[1] The proposition (3) has been removed.

The set PredSym is defined by:

(Def. 5) $\mathrm{PredSym} = \{\langle k, l\rangle : 7 \leq k\}$.

One can verify the following observations:

* BoundVar is non empty,
* FixedVar is non empty,
* FreeVar is non empty, and
* PredSym is non empty.

The following proposition is true

(10)[2] $\mathrm{PredSym} \subseteq [:\mathbb{N}, \mathbb{N}:]$.

A predicate symbol is an element of PredSym.

Let P be an element of PredSym. The functor $\mathrm{Arity}(P)$ yields a natural number and is defined by:

(Def. 6) $P_{\mathbf{1}} = 7 + \mathrm{Arity}(P)$.

In the sequel P denotes a predicate symbol.

Let us consider k. The functor $\mathrm{PredSym}_k$ yielding a subset of PredSym is defined by:

(Def. 7) $\mathrm{PredSym}_k = \{P : \mathrm{Arity}(P) = k\}$.

Let us consider k. Note that $\mathrm{PredSym}_k$ is non empty.

A bound variable is an element of BoundVar. A fixed variable is an element of FixedVar. A free variable is an element of FreeVar. Let us consider k. A k-ary predicate symbol is an element of $\mathrm{PredSym}_k$.

Let k be a natural number. A finite sequence of elements of Var is said to be a list of variables of the length k if:

(Def. 8) $\mathrm{len}\,\mathrm{it} = k$.

Let D be a set. We say that D is closed if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) D is a subset of $[:\mathbb{N}, \mathbb{N}:]^*$,

(ii) for every natural number k and for every k-ary predicate symbol p and for every list of variables l_1 of the length k holds $\langle p\rangle ^\frown l_1 \in D$,

(iii) $\langle\langle 0, 0\rangle\rangle \in D$,

(iv) for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $\langle\langle 1, 0\rangle\rangle ^\frown p \in D$,

(v) for all finite sequences p, q of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ and $q \in D$ holds $\langle\langle 2, 0\rangle\rangle ^\frown p ^\frown q \in D$, and

(vi) for every bound variable x and for every finite sequence p of elements of $[:\mathbb{N}, \mathbb{N}:]$ such that $p \in D$ holds $\langle\langle 3, 0\rangle\rangle ^\frown \langle x\rangle ^\frown p \in D$.

The non empty set WFF is defined by:

(Def. 10) WFF is closed and for every non empty set D such that D is closed holds $\mathrm{WFF} \subseteq D$.

We now state the proposition

(21)[3] WFF is closed.

[2] The propositions (5)–(9) have been removed.

[3] The propositions (11)–(20) have been removed.

A formula is an element of WFF.

Let P be a predicate symbol and let l be a finite sequence of elements of Var. Let us assume that $\mathrm{Arity}(P) = \mathrm{len}\, l$. The functor $P[l]$ yielding an element of WFF is defined by:

(Def. 11) $P[l] = \langle P \rangle \frown l$.

The following proposition is true

(23)[4] Let k be a natural number, p be a k-ary predicate symbol, and l_1 be a list of variables of the length k. Then $p[l_1] = \langle p \rangle \frown l_1$.

Let p be an element of WFF. The functor ${}^{@}p$ yields a finite sequence of elements of $[: \mathbb{N}, \mathbb{N} :]$ and is defined by:

(Def. 12) ${}^{@}p = p$.

The formula VERUM is defined as follows:

(Def. 13) $\mathrm{VERUM} = \langle \langle 0, 0 \rangle \rangle$.

Let p be an element of WFF. The functor $\neg p$ yielding a formula is defined as follows:

(Def. 14) $\neg p = \langle \langle 1, 0 \rangle \rangle \frown ({}^{@}p)$.

Let q be an element of WFF. The functor $p \wedge q$ yielding a formula is defined by:

(Def. 15) $p \wedge q = \langle \langle 2, 0 \rangle \rangle \frown ({}^{@}p) \frown ({}^{@}q)$.

Let x be a bound variable and let p be an element of WFF. The functor $\forall_x p$ yields a formula and is defined by:

(Def. 16) $\forall_x p = \langle \langle 3, 0 \rangle \rangle \frown \langle x \rangle \frown ({}^{@}p)$.

The scheme *QC Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every element F of WFF holds $\mathcal{P}[F]$

provided the parameters meet the following conditions:

- Let k be a natural number, P be a k-ary predicate symbol, and l_1 be a list of variables of the length k. Then $\mathcal{P}[P[l_1]]$,
- $\mathcal{P}[\mathrm{VERUM}]$,
- For every element p of WFF such that $\mathcal{P}[p]$ holds $\mathcal{P}[\neg p]$,
- For all elements p, q of WFF such that $\mathcal{P}[p]$ and $\mathcal{P}[q]$ holds $\mathcal{P}[p \wedge q]$, and
- For every bound variable x and for every element p of WFF such that $\mathcal{P}[p]$ holds $\mathcal{P}[\forall_x p]$.

Let F be an element of WFF. We say that F is atomic if and only if the condition (Def. 17) is satisfied.

(Def. 17) There exists a natural number k and there exists a k-ary predicate symbol p and there exists a list of variables l_1 of the length k such that $F = p[l_1]$.

We say that F is negative if and only if:

(Def. 18) There exists an element p of WFF such that $F = \neg p$.

We say that F is conjunctive if and only if:

(Def. 19) There exist elements p, q of WFF such that $F = p \wedge q$.

We say that F is universal if and only if:

(Def. 20) There exists a bound variable x and there exists an element p of WFF such that $F = \forall_x p$.

One can prove the following propositions:

[4] The proposition (22) has been removed.

(33)[5] For every element F of WFF holds $F = \mathrm{VERUM}$ or F is atomic, negative, conjunctive, and universal.

(34) For every element F of WFF holds $1 \leq \mathrm{len}({}^{@}F)$.

(35) For every natural number k and for every k-ary predicate symbol P holds $\mathrm{Arity}(P) = k$.

In the sequel F, G are elements of WFF and s is a finite sequence.
The following propositions are true:

(36)(i) If $({}^{@}F)(1)_{\mathbf{1}} = 0$, then $F = \mathrm{VERUM}$,

(ii) if $({}^{@}F)(1)_{\mathbf{1}} = 1$, then F is negative,

(iii) if $({}^{@}F)(1)_{\mathbf{1}} = 2$, then F is conjunctive,

(iv) if $({}^{@}F)(1)_{\mathbf{1}} = 3$, then F is universal, and

(v) if there exists a natural number k such that $({}^{@}F)(1)$ is a k-ary predicate symbol, then F is atomic.

(37) If ${}^{@}F = ({}^{@}G) \frown s$, then ${}^{@}F = {}^{@}G$.

Let F be an element of WFF. Let us assume that F is atomic. The functor $\mathrm{PredSym}(F)$ yielding a predicate symbol is defined by the condition (Def. 21).

(Def. 21) There exists a natural number k and there exists a list of variables l_1 of the length k and there exists a k-ary predicate symbol P such that $\mathrm{PredSym}(F) = P$ and $F = P[l_1]$.

Let F be an element of WFF. Let us assume that F is atomic. The functor $\mathrm{Args}(F)$ yielding a finite sequence of elements of Var is defined by the condition (Def. 22).

(Def. 22) There exists a natural number k and there exists a k-ary predicate symbol P and there exists a list of variables l_1 of the length k such that $\mathrm{Args}(F) = l_1$ and $F = P[l_1]$.

Let F be an element of WFF. Let us assume that F is negative. The functor $\mathrm{Arg}(F)$ yields a formula and is defined as follows:

(Def. 23) $F = \neg \mathrm{Arg}(F)$.

Let F be an element of WFF. Let us assume that F is conjunctive. The functor $\mathrm{LeftArg}(F)$ yielding a formula is defined as follows:

(Def. 24) There exists an element q of WFF such that $F = \mathrm{LeftArg}(F) \wedge q$.

Let F be an element of WFF. Let us assume that F is conjunctive. The functor $\mathrm{RightArg}(F)$ yields a formula and is defined as follows:

(Def. 25) There exists an element p of WFF such that $F = p \wedge \mathrm{RightArg}(F)$.

Let F be an element of WFF. Let us assume that F is universal. The functor $\mathrm{Bound}(F)$ yielding a bound variable is defined as follows:

(Def. 26) There exists an element p of WFF such that $F = \forall_{\mathrm{Bound}(F)} p$.

The functor $\mathrm{Scope}(F)$ yielding a formula is defined by:

(Def. 27) There exists a bound variable x such that $F = \forall_x \mathrm{Scope}(F)$.

In the sequel p denotes an element of WFF.
The following propositions are true:

(45)[6] If p is negative, then $\mathrm{len}({}^{@}\mathrm{Arg}(p)) < \mathrm{len}({}^{@}p)$.

[5] The propositions (24)–(32) have been removed.
[6] The propositions (38)–(44) have been removed.

(46) If p is conjunctive, then $\mathrm{len}({}^{@}\mathrm{LeftArg}(p)) < \mathrm{len}({}^{@}p)$ and $\mathrm{len}({}^{@}\mathrm{RightArg}(p)) < \mathrm{len}({}^{@}p)$.

(47) If p is universal, then $\mathrm{len}({}^{@}\mathrm{Scope}(p)) < \mathrm{len}({}^{@}p)$.

The scheme *QC Ind2* concerns a unary predicate $\mathcal{P}$, and states that:
For every element p of WFF holds $\mathcal{P}[p]$
provided the parameters meet the following condition:

- Let p be an element of WFF. Then
  - (i) if p is atomic, then $\mathcal{P}[p]$,
  - (ii) $\mathcal{P}[\mathrm{VERUM}]$,
  - (iii) if p is negative and $\mathcal{P}[\mathrm{Arg}(p)]$, then $\mathcal{P}[p]$,
  - (iv) if p is conjunctive and $\mathcal{P}[\mathrm{LeftArg}(p)]$ and $\mathcal{P}[\mathrm{RightArg}(p)]$, then $\mathcal{P}[p]$, and
  - (v) if p is universal and $\mathcal{P}[\mathrm{Scope}(p)]$, then $\mathcal{P}[p]$.

In the sequel F denotes an element of WFF.
Next we state three propositions:

(48) For every natural number k and for every k-ary predicate symbol P holds $P_{\mathbf{1}} \neq 0$ and $P_{\mathbf{1}} \neq 1$ and $P_{\mathbf{1}} \neq 2$ and $P_{\mathbf{1}} \neq 3$.

(49)(i) $({}^{@}\mathrm{VERUM})(1)_{\mathbf{1}} = 0$,

(ii) if F is atomic, then there exists a natural number k such that $({}^{@}F)(1)$ is a k-ary predicate symbol,

(iii) if F is negative, then $({}^{@}F)(1)_{\mathbf{1}} = 1$,

(iv) if F is conjunctive, then $({}^{@}F)(1)_{\mathbf{1}} = 2$, and

(v) if F is universal, then $({}^{@}F)(1)_{\mathbf{1}} = 3$.

(50) If F is atomic, then $({}^{@}F)(1)_{\mathbf{1}} \neq 0$ and $({}^{@}F)(1)_{\mathbf{1}} \neq 1$ and $({}^{@}F)(1)_{\mathbf{1}} \neq 2$ and $({}^{@}F)(1)_{\mathbf{1}} \neq 3$.

In the sequel p denotes an element of WFF.
The following proposition is true

(51)(i) VERUM is not atomic and VERUM is not negative and VERUM is not conjunctive and VERUM is not universal, and

(ii) there exists no p which is atomic, negative, atomic, conjunctive, atomic, universal, negative, conjunctive, negative, universal, conjunctive, and universal.

The scheme *QC Func Ex* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{A}$, a binary functor $\mathcal{H}$ yielding an element of $\mathcal{A}$, and a binary functor $\mathcal{I}$ yielding an element of $\mathcal{A}$, and states that:
There exists a function F from WFF into $\mathcal{A}$ such that
(i) $F(\mathrm{VERUM}) = \mathcal{B}$, and
(ii) for every element p of WFF holds if p is atomic, then $F(p) = \mathcal{F}(p)$ and if p is negative, then $F(p) = \mathcal{G}(F(\mathrm{Arg}(p)))$ and if p is conjunctive, then $F(p) = \mathcal{H}(F(\mathrm{LeftArg}(p)), F(\mathrm{RightArg}(p)))$ and if p is universal, then $F(p) = \mathcal{I}(p, F(\mathrm{Scope}(p)))$
for all values of the parameters.

In the sequel k denotes a natural number.

Let l_1 be a finite sequence of elements of Var. The functor $\mathrm{snb}(l_1)$ yields an element of 2^{BoundVar} and is defined by:

(Def. 28) $\mathrm{snb}(l_1) = \{l_1(k) : 1 \leq k \wedge k \leq \mathrm{len}\, l_1 \wedge l_1(k) \in \mathrm{BoundVar}\}$.

Let b be a bound variable. Then $\{b\}$ is an element of 2^{BoundVar}.

Let X, Y be elements of 2^{BoundVar}. Then $X \cup Y$ is an element of 2^{BoundVar}. Then $X \setminus Y$ is an element of 2^{BoundVar}.

In the sequel k denotes a natural number.

Let p be a formula. The functor $\mathrm{snb}(p)$ yields an element of 2^{BoundVar} and is defined by the condition (Def. 29).

(Def. 29) There exists a function F from WFF into 2^{BoundVar} such that

(i) $\mathrm{snb}(p) = F(p)$, and

(ii) for every element p of WFF holds $F(\mathrm{VERUM}) = \emptyset$ and if p is atomic, then $F(p) = \{\mathrm{Args}(p)(k) : 1 \leq k \wedge k \leq \mathrm{len}\,\mathrm{Args}(p) \wedge \mathrm{Args}(p)(k) \in \mathrm{BoundVar}\}$ and if p is negative, then $F(p) = F(\mathrm{Arg}(p))$ and if p is conjunctive, then $F(p) = F(\mathrm{LeftArg}(p)) \cup F(\mathrm{RightArg}(p))$ and if p is universal, then $F(p) = F(\mathrm{Scope}(p)) \setminus \{\mathrm{Bound}(p)\}$.

Let p be a formula. We say that p is closed if and only if:

(Def. 30) $\mathrm{snb}(p) = \emptyset$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[6] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/enumset1.html`.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[8] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[9] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[11] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[12] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received August 8, 1989

Published January 2, 2004