JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Product of Families of Groups and Vector Spaces

Anna Lango
Warsaw University
Białystok

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Summary. In the first section we present properties of fields and Abelian groups in terms of commutativity, associativity, etc. Next, we are concerned with operations on n-tuples on some set which are generalization of operations on this set. It is used in third section to introduce the n-power of a group and the n-power of a field. Besides, we introduce a concept of indexed family of binary (unary) operations over some indexed family of sets and a product of such families which is binary (unary) operation on a product of family sets. We use that product in the last section to introduce the product of a finite sequence of Abelian groups.

MML Identifier: PRVECT_1.

WWW: http://mizar.org/JFM/Vol4/prvect_1.html

The articles [13], [7], [16], [1], [17], [5], [6], [4], [11], [15], [10], [8], [3], [14], [2], [12], and [9] provide the notation and terminology for this paper.

1. ABELIAN GROUPS AND FIELDS

In this paper G is an Abelian add-associative right complementable right zeroed non empty loop structure.

One can prove the following two propositions:

(3)[1] The zero of G is a unity w.r.t. the addition of G.

(4) For every Abelian group G holds comp G is an inverse operation w.r.t. the addition of G.

In the sequel G_1 is a non empty loop structure.

One can prove the following proposition

(5) Suppose that

(i) the addition of G_1 is commutative and associative,

(ii) the zero of G_1 is a unity w.r.t. the addition of G_1, and

(iii) comp G_1 is an inverse operation w.r.t. the addition of G_1.

Then G_1 is an Abelian group.

In the sequel F denotes a field.

Next we state two propositions:

(10)[2] The zero of F is a unity w.r.t. the addition of F.

(11) The unity of F is a unity w.r.t. the multiplication of F.

[1] The propositions (1) and (2) have been removed.
[2] The propositions (6)–(9) have been removed.

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2. The n-Product of a Binary and a Unary Operation

For simplicity, we follow the rules: F denotes a field, n denotes a natural number, D denotes a non empty set, d denotes an element of D, B denotes a binary operation on D, and C denotes a unary operation on D.

Let us consider D, n, let F be a binary operation on D, and let x, y be elements of D^n. Then $F^\circ(x, y)$ is an element of D^n.

Let D be a non empty set, let F be a binary operation on D, and let n be a natural number. The functor $\pi^n F$ yields a binary operation on D^n and is defined by:

(Def. 1) For all elements x, y of D^n holds $(\pi^n F)(x, y) = F^\circ(x, y)$.

Let us consider D, let F be a unary operation on D, and let us consider n. The functor $\pi^n F$ yielding a unary operation on D^n is defined by:

(Def. 2) For every element x of D^n holds $(\pi^n F)(x) = F \cdot x$.

Let D be a non empty set, let us consider n, and let x be an element of D. Then $n \mapsto x$ is an element of D^n. We introduce $n \dot{\longmapsto} x$ as a synonym of $n \mapsto x$.

One can prove the following four propositions:

(14)[3] If B is commutative, then $\pi^n B$ is commutative.

(15) If B is associative, then $\pi^n B$ is associative.

(16) If d is a unity w.r.t. B, then $n \dot{\longmapsto} d$ is a unity w.r.t. $\pi^n B$.

(17) If B is associative and has a unity and C is an inverse operation w.r.t. B, then $\pi^n C$ is an inverse operation w.r.t. $\pi^n B$.

3. The n-Power of a Group and of a Field

Let F be a non empty loop structure and let us consider n. Let us assume that F is Abelian, add-associative, right zeroed, and right complementable. The functor F^n yields a strict Abelian group and is defined as follows:

(Def. 3) $F^n = \langle$(the carrier of $F)^n, \pi^n$(the addition of F),($n \dot{\longmapsto}$the zero of F **qua** element of (the carrier of $F)^n)\rangle$.

Let F be an Abelian group and let us consider n. Observe that F^n is non empty.

Let us consider F, n. The functor $\cdot^n_F$ yielding a function from [:the carrier of F, (the carrier of $F)^n$:] into (the carrier of $F)^n$ is defined by the condition (Def. 4).

(Def. 4) Let x be an element of F and v be an element of (the carrier of $F)^n$. Then $(\cdot^n_F)(x, v) =$ (the multiplication of $F)^\circ(x, v)$.

Let us consider F, n. The n-dimension vector space over F yields a strict vector space structure over F and is defined by the conditions (Def. 5).

(Def. 5)(i) The loop structure of the n-dimension vector space over $F = F^n$, and

(ii) the left multiplication of the n-dimension vector space over $F = \cdot^n_F$.

Let us consider F, n. Observe that the n-dimension vector space over F is non empty.

For simplicity, we use the following convention: D denotes a non empty set, H, G denote binary operations on D, d denotes an element of D, and t_1, t_2 denote elements of D^n.

One can prove the following proposition

(18) If H is distributive w.r.t. G, then $H^\circ(d, G^\circ(t_1, t_2)) = G^\circ(H^\circ(d, t_1), H^\circ(d, t_2))$.

Let D be a non empty set, let n be a natural number, let F be a binary operation on D, let x be an element of D, and let v be an element of D^n. Then $F^\circ(x, v)$ is an element of D^n.

Let us consider F, n. Note that the n-dimension vector space over F is vector space-like.

[3] The propositions (12) and (13) have been removed.

4. Sequences of Non-empty Sets

In the sequel x is a set.

Let us observe that there exists a finite sequence which is non empty and non-empty.

Let f be a non-empty function. One can verify that $\prod f$ is functional and non empty.

A sequence of non empty sets is a non empty non-empty finite sequence.

Let a be a non empty function. One can verify that $\mathrm{dom}\, a$ is non empty.

The scheme *NEFinSeqLambda* deals with a non empty finite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a non empty finite sequence p such that $\mathrm{len}\, p = \mathrm{len}\, \mathcal{A}$ and for every element i of $\mathrm{dom}\, \mathcal{A}$ holds $p(i) = \mathcal{F}(i)$

for all values of the parameters.

Let a be a non-empty non empty function and let i be an element of $\mathrm{dom}\, a$. One can verify that $a(i)$ is non empty.

Let a be a non-empty non empty function, let f be an element of $\prod a$, and let i be an element of $\mathrm{dom}\, a$. Then $f(i)$ is an element of $a(i)$.

5. The Product of Families of Operations

In the sequel a is a sequence of non empty sets, i is an element of $\mathrm{dom}\, a$, and p is a finite sequence.

Let a be a non-empty non empty function. A function is called a family of binary operations of a if:

(Def. 8)[4] $\mathrm{dom}\, \mathrm{it} = \mathrm{dom}\, a$ and for every element i of $\mathrm{dom}\, a$ holds $\mathrm{it}(i)$ is a binary operation on $a(i)$.

A function is called a family of unary operations of a if:

(Def. 9) $\mathrm{dom}\, \mathrm{it} = \mathrm{dom}\, a$ and for every element i of $\mathrm{dom}\, a$ holds $\mathrm{it}(i)$ is a unary operation on $a(i)$.

Let us consider a. Note that every family of binary operations of a is finite sequence-like and every family of unary operations of a is finite sequence-like.

Next we state two propositions:

(19) p is a family of binary operations of a if and only if $\mathrm{len}\, p = \mathrm{len}\, a$ and for every i holds $p(i)$ is a binary operation on $a(i)$.

(20) p is a family of unary operations of a if and only if $\mathrm{len}\, p = \mathrm{len}\, a$ and for every i holds $p(i)$ is a unary operation on $a(i)$.

Let us consider a, let b be a family of binary operations of a, and let us consider i. Then $b(i)$ is a binary operation on $a(i)$.

Let us consider a, let u be a family of unary operations of a, and let us consider i. Then $u(i)$ is a unary operation on $a(i)$.

Let F be a functional non empty set, let u be a unary operation on F, and let f be an element of F. Then $u(f)$ is an element of F.

One can prove the following two propositions:

(21) Let d, d' be unary operations on $\prod a$. Suppose that for every element f of $\prod a$ and for every element i of $\mathrm{dom}\, a$ holds $d(f)(i) = d'(f)(i)$. Then $d = d'$.

(22) For every family u of unary operations of a holds $\mathrm{dom}_\kappa u(\kappa) = a$ and $\prod(\mathrm{rng}_\kappa u(\kappa)) \subseteq \prod a$.

Let us consider a and let u be a family of unary operations of a. Then $\mathrm{Frege}(u)$ is a unary operation on $\prod a$.

We now state the proposition

(23) Let u be a family of unary operations of a, f be an element of $\prod a$, and i be an element of $\mathrm{dom}\, a$. Then $(\mathrm{Frege}(u))(f)(i) = u(i)(f(i))$.

[4] The definitions (Def. 6) and (Def. 7) have been removed.

Let F be a functional non empty set, let b be a binary operation on F, and let f, g be elements of F. Then $b(f, g)$ is an element of F.

We now state the proposition

(24) Let d, d' be binary operations on $\prod a$. Suppose that for all elements f, g of $\prod a$ and for every element i of $\mathrm{dom}\, a$ holds $d(f, g)(i) = d'(f, g)(i)$. Then $d = d'$.

In the sequel i denotes an element of $\mathrm{dom}\, a$.

Let us consider a and let b be a family of binary operations of a. The functor $\prod^\circ b$ yields a binary operation on $\prod a$ and is defined as follows:

(Def. 10) For all elements f, g of $\prod a$ and for every element i of $\mathrm{dom}\, a$ holds $(\prod^\circ b)(f, g)(i) = b(i)(f(i), g(i))$.

Next we state four propositions:

(25) For every family b of binary operations of a such that for every i holds $b(i)$ is commutative holds $\prod^\circ b$ is commutative.

(26) For every family b of binary operations of a such that for every i holds $b(i)$ is associative holds $\prod^\circ b$ is associative.

(27) Let b be a family of binary operations of a and f be an element of $\prod a$. If for every i holds $f(i)$ is a unity w.r.t. $b(i)$, then f is a unity w.r.t. $\prod^\circ b$.

(28) Let b be a family of binary operations of a and u be a family of unary operations of a. Suppose that for every i holds $u(i)$ is an inverse operation w.r.t. $b(i)$ and $b(i)$ has a unity. Then $\mathrm{Frege}(u)$ is an inverse operation w.r.t. $\prod^\circ b$.

6. The Product of Families of Groups

Let F be a function. We say that F is Abelian group yielding if and only if:

(Def. 11) If $x \in \mathrm{rng}\, F$, then x is an Abelian group.

Let us note that there exists a finite sequence which is non empty and Abelian group yielding.

A sequence of groups is a non empty Abelian group yielding finite sequence.

Let g be a sequence of groups and let i be an element of $\mathrm{dom}\, g$. Then $g(i)$ is an Abelian group.

Let g be a sequence of groups. The functor $\overline{g}$ yields a sequence of non empty sets and is defined by:

(Def. 12) $\mathrm{len}\, \overline{g} = \mathrm{len}\, g$ and for every element j of $\mathrm{dom}\, g$ holds $\overline{g}(j) =$ the carrier of $g(j)$.

In the sequel g is a sequence of groups and i is an element of $\mathrm{dom}\, \overline{g}$.

Let us consider g, i. Then $g(i)$ is an Abelian group.

Let us consider g. The functor $\langle +_{g_i} \rangle_i$ yielding a family of binary operations of $\overline{g}$ is defined by:

(Def. 13) $\mathrm{len}(\langle +_{g_i} \rangle_i) = \mathrm{len}\, \overline{g}$ and for every i holds $\langle +_{g_i} \rangle_i(i) =$ the addition of $g(i)$.

The functor $\langle -_{g_i} \rangle_i$ yielding a family of unary operations of $\overline{g}$ is defined as follows:

(Def. 14) $\mathrm{len}(\langle -_{g_i} \rangle_i) = \mathrm{len}\, \overline{g}$ and for every i holds $\langle -_{g_i} \rangle_i(i) = \mathrm{comp}\, g(i)$.

The functor $\langle 0_{g_i} \rangle_i$ yields an element of $\prod \overline{g}$ and is defined as follows:

(Def. 15) For every i holds $\langle 0_{g_i} \rangle_i(i) =$ the zero of $g(i)$.

Let G be a sequence of groups. The functor $\prod G$ yielding a strict Abelian group is defined as follows:

(Def. 16) $\prod G = \langle \prod \overline{G}, \prod^\circ(\langle +_{G_i} \rangle_i), \langle 0_{G_i} \rangle_i \rangle$.

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Received December 29, 1992

Published January 2, 2004