JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Topological Spaces and Continuous Functions[1]

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Summary. The paper contains a definition of topological space. The following notions are defined: point of topological space, subset of topological space, subspace of topological space, and continuous function.

MML Identifier: `PRE_TOPC`.

WWW: `http://mizar.org/JFM/Vol1/pre_topc.html`

The articles [4], [2], [5], [6], [1], and [3] provide the notation and terminology for this paper.

We consider topological structures as extensions of 1-sorted structure as systems

⟨ a carrier, a topology ⟩,

where the carrier is a set and the topology is a family of subsets of the carrier.

In the sequel T is a topological structure.

Let I_1 be a topological structure. We say that I_1 is topological space-like if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) The carrier of $I_1 \in$ the topology of I_1,

(ii) for every family a of subsets of I_1 such that $a \subseteq$ the topology of I_1 holds $\bigcup a \in$ the topology of I_1, and

(iii) for all subsets a, b of I_1 such that $a \in$ the topology of I_1 and $b \in$ the topology of I_1 holds $a \cap b \in$ the topology of I_1.

Let us note that there exists a topological structure which is non empty, strict, and topological space-like.

A topological space is a topological space-like topological structure.

Let S be a 1-sorted structure. A point of S is an element of S.

In the sequel G_1 is a topological space.

Next we state the proposition

(5)[1] $\emptyset \in$ the topology of G_1.

Let T be a 1-sorted structure. The functor $\emptyset_T$ yields a subset of T and is defined as follows:

(Def. 2) $\emptyset_T = \emptyset$.

The functor Ω_T yielding a subset of T is defined by:

(Def. 3) $\Omega_T =$ the carrier of T.

[1] Supported by RPBP.III-24.C1.

[1] The propositions (1)–(4) have been removed.

Let T be a 1-sorted structure. Observe that $\emptyset_T$ is empty.
One can prove the following proposition

(12)[2] For every 1-sorted structure T holds $\Omega_T =$ the carrier of T.

Let T be a non empty 1-sorted structure. Note that Ω_T is non empty.
The following propositions are true:

(13) For every non empty 1-sorted structure T and for every point p of T holds $p \in \Omega_T$.

(14) For every 1-sorted structure T and for every subset P of T holds $P \subseteq \Omega_T$.

(15) For every 1-sorted structure T and for every subset P of T holds $P \cap \Omega_T = P$.

(16) For every 1-sorted structure T and for every set A such that $A \subseteq \Omega_T$ holds A is a subset of T.

(17) For every 1-sorted structure T and for every subset P of T holds $P^{\rm c} = \Omega_T \setminus P$.

(18) For every 1-sorted structure T and for every subset P of T holds $P \cup P^{\rm c} = \Omega_T$.

(19) For every 1-sorted structure T and for all subsets P, Q of T holds $P \subseteq Q$ iff $Q^{\rm c} \subseteq P^{\rm c}$.

(20) For every 1-sorted structure T and for every subset P of T holds $P = (P^{\rm c})^{\rm c}$.

(21) For every 1-sorted structure T and for all subsets P, Q of T holds $P \subseteq Q^{\rm c}$ iff P misses Q.

(22) For every 1-sorted structure T and for every subset P of T holds $\Omega_T \setminus (\Omega_T \setminus P) = P$.

(23) For every 1-sorted structure T and for every subset P of T holds $P \neq \Omega_T$ iff $\Omega_T \setminus P \neq \emptyset$.

(24) For every 1-sorted structure T and for all subsets P, Q of T such that $\Omega_T \setminus P = Q$ holds $\Omega_T = P \cup Q$.

(25) For every 1-sorted structure T and for all subsets P, Q of T such that $\Omega_T = P \cup Q$ and P misses Q holds $Q = \Omega_T \setminus P$.

(26) For every 1-sorted structure T and for every subset P of T holds P misses $P^{\rm c}$.

(27) For every 1-sorted structure T holds $\Omega_T = (\emptyset_T)^{\rm c}$.

Let T be a topological structure and let P be a subset of T. We say that P is open if and only if:

(Def. 5)[3] $P \in$ the topology of T.

Let T be a topological structure and let P be a subset of T. We say that P is closed if and only if:

(Def. 6) $\Omega_T \setminus P$ is open.

Let T be a 1-sorted structure and let F be a family of subsets of T. Then $\bigcup F$ is a subset of T.
Let T be a 1-sorted structure and let F be a family of subsets of T. Then $\bigcap F$ is a subset of T.
Let T be a 1-sorted structure and let F be a family of subsets of T. We say that F is a cover of T if and only if:

(Def. 8)[4] $\Omega_T = \bigcup F$.

Let T be a topological structure. A topological structure is said to be a subspace of T if it satisfies the conditions (Def. 9).

[2] The propositions (6)–(11) have been removed.
[3] The definition (Def. 4) has been removed.
[4] The definition (Def. 7) has been removed.

(Def. 9)(i) $\Omega_{\rm it} \subseteq \Omega_T$, and

(ii) for every subset P of it holds $P \in$ the topology of it iff there exists a subset Q of T such that $Q \in$ the topology of T and $P = Q \cap \Omega_{\rm it}$.

Let T be a topological structure. Observe that there exists a subspace of T which is strict.

Let T be a non empty topological structure. Observe that there exists a subspace of T which is strict and non empty.

The scheme *SubFamExS* deals with a topological structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let T be a topological space. One can check that every subspace of T is topological space-like.

Let T be a topological structure and let P be a subset of T. The functor $T \restriction P$ yielding a strict subspace of T is defined as follows:

(Def. 10) $\Omega_{T \restriction P} = P$.

Let T be a non empty topological structure and let P be a non empty subset of T. Note that $T \restriction P$ is non empty.

Let T be a topological space. Note that there exists a subspace of T which is topological space-like and strict.

Let T be a topological space and let P be a subset of T. Note that $T \restriction P$ is topological space-like.

Let S, T be 1-sorted structures. A map from S into T is a function from the carrier of S into the carrier of T.

Let S, T be 1-sorted structures, let f be a function from the carrier of S into the carrier of T, and let P be a set. Then $f^\circ P$ is a subset of T.

Let S, T be 1-sorted structures, let f be a function from the carrier of S into the carrier of T, and let P be a set. Then $f^{-1}(P)$ is a subset of S.

Let S, T be topological structures and let f be a map from S into T. We say that f is continuous if and only if:

(Def. 12)[5] For every subset P_1 of T such that P_1 is closed holds $f^{-1}(P_1)$ is closed.

The scheme *TopAbstr* deals with a topological structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a subset P of $\mathcal{A}$ such that for every set x such that $x \in$ the carrier of $\mathcal{A}$ holds $x \in P$ iff $\mathcal{P}[x]$

for all values of the parameters.

Next we state three propositions:

(39)[6] For every subspace X' of T holds every subset of X' is a subset of T.

(41)[7] For every subset A of T such that $A \neq \emptyset_T$ there exists a point x of T such that $x \in A$.

(42) $\Omega_{(G_1)}$ is closed.

Let T be a topological space. One can verify that Ω_T is closed.

Let T be a topological space. Note that there exists a subset of T which is closed.

Let T be a non empty topological space. One can verify that there exists a subset of T which is non empty and closed.

We now state two propositions:

(43) Let X' be a subspace of T and B be a subset of X'. Then B is closed if and only if there exists a subset C of T such that C is closed and $C \cap \Omega_{X'} = B$.

[5] The definition (Def. 11) has been removed.

[6] The propositions (28)–(38) have been removed.

[7] The proposition (40) has been removed.

(44) Let F be a family of subsets of G_1. Suppose that for every subset A of G_1 such that $A \in F$ holds A is closed. Then $\bigcap F$ is closed.

Let G_1 be a topological structure and let A be a subset of G_1. The functor $\overline{A}$ yields a subset of G_1 and is defined by the condition (Def. 13).

(Def. 13) Let p be a set. Suppose $p \in$ the carrier of G_1. Then $p \in \overline{A}$ if and only if for every subset G of G_1 such that G is open holds if $p \in G$, then A meets G.

We now state a number of propositions:

(45) Let A be a subset of T and p be a set. Suppose $p \in$ the carrier of T. Then $p \in \overline{A}$ if and only if for every subset C of T such that C is closed holds if $A \subseteq C$, then $p \in C$.

(46) Let A be a subset of G_1. Then there exists a family F of subsets of G_1 such that for every subset C of G_1 holds $C \in F$ iff C is closed and $A \subseteq C$ and $\overline{A} = \bigcap F$.

(47) For every subspace X' of T and for every subset A of T and for every subset A_1 of X' such that $A = A_1$ holds $\overline{A_1} = \overline{A} \cap \Omega_{X'}$.

(48) For every subset A of T holds $A \subseteq \overline{A}$.

(49) For all subsets A, B of T such that $A \subseteq B$ holds $\overline{A} \subseteq \overline{B}$.

(50) For all subsets A, B of G_1 holds $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(51) For all subsets A, B of T holds $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

(52) Let A be a subset of T. Then

(i) if A is closed, then $\overline{A} = A$, and

(ii) if T is topological space-like and $\overline{A} = A$, then A is closed.

(53) Let A be a subset of T. Then

(i) if A is open, then $\overline{\Omega_T \setminus A} = \Omega_T \setminus A$, and

(ii) if T is topological space-like and $\overline{\Omega_T \setminus A} = \Omega_T \setminus A$, then A is open.

(54) Let A be a subset of T and p be a point of T. Then $p \in \overline{A}$ if and only if the following conditions are satisfied:

(i) T is non empty, and

(ii) for every subset G of T such that G is open holds if $p \in G$, then A meets G.

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Received April 14, 1989

Published January 2, 2004