JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Products of Many Sorted Algebras

Beata Madras
Warsaw University
Białystok

**Summary.** Product of two many sorted universal algebras and product of family of many sorted universal algebras are defined in this article. Operations on functions, such that commute, Frege, are also introduced.

MML Identifier: PRALG_2.

WWW: http://mizar.org/JFM/Vol6/pralg_2.html

The articles [11], [7], [16], [17], [5], [6], [12], [8], [13], [9], [1], [3], [2], [4], [14], [10], and [15] provide the notation and terminology for this paper.

## 1. Preliminaries

We follow the rules: $I$, $J$, $i$, $j$, $x$ denote sets, $A$, $B$ denote many sorted sets indexed by $I$, and $S$ denotes a non empty many sorted signature.

Let $I_1$ be a set. We say that $I_1$ has common domain if and only if:

(Def. 1) For all functions $f$, $g$ such that $f \in I_1$ and $g \in I_1$ holds $\operatorname{dom} f = \operatorname{dom} g$.

Let us note that there exists a set which is functional and non empty and has common domain.

The following proposition is true

(1) $\{\emptyset\}$ is a functional set with common domain.

Let $X$ be a functional set with common domain. The functor $\mathrm{DOM}(X)$ yielding a set is defined as follows:

(Def. 2)(i) For every function $x$ such that $x \in X$ holds $\mathrm{DOM}(X) = \operatorname{dom} x$ if $X \neq \emptyset$,

(ii) $\mathrm{DOM}(X) = \emptyset$, otherwise.

Let $X$ be a functional non empty set with common domain. We see that the element of $X$ is a many sorted set indexed by $\mathrm{DOM}(X)$.

The following proposition is true

(2) For every functional set $X$ with common domain such that $X = \{\emptyset\}$ holds $\mathrm{DOM}(X) = \emptyset$.

Let $I$ be a set and let $M$ be a non-empty many sorted set indexed by $I$. One can verify that $\prod M$ is functional and non empty and has common domain.

 

## 2. Operations on Functions

The scheme *LambdaDMS* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a many sorted set $X$ indexed by $\mathcal{A}$ such that for every element $d$ of $\mathcal{A}$ holds $X(d) = \mathcal{F}(d)$

for all values of the parameters.

Let $f$ be a function. The functor $\mathrm{commute}(f)$ yields a function yielding function and is defined as follows:

(Def. 5)[1] $\mathrm{commute}(f) = \mathrm{curry}' \mathrm{uncurry} f$.

The following propositions are true:

(3) For every function $f$ and for every set $x$ such that $x \in \mathrm{dom}\,\mathrm{commute}(f)$ holds $(\mathrm{commute}(f))(x)$ is a function.

(4) For all sets $A, B, C$ and for every function $f$ such that $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$ holds $\mathrm{commute}(f) \in (C^A)^B$.

(5) Let $A, B, C$ be sets and $f$ be a function. Suppose $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$. Let $g, h$ be functions and $x, y$ be sets. Suppose $x \in A$ and $y \in B$ and $f(x) = g$ and $(\mathrm{commute}(f))(y) = h$. Then $h(x) = g(y)$ and $\mathrm{dom}\,h = A$ and $\mathrm{dom}\,g = B$ and $\mathrm{rng}\,h \subseteq C$ and $\mathrm{rng}\,g \subseteq C$.

(6) For all sets $A, B, C$ and for every function $f$ such that $A \neq \emptyset$ and $B \neq \emptyset$ and $f \in (C^B)^A$ holds $\mathrm{commute}(\mathrm{commute}(f)) = f$.

(7) $\mathrm{commute}(\emptyset) = \emptyset$.

Let $F$ be a function. The functor $\blacksquare\mathrm{commute}(F)$ yields a function and is defined by the conditions (Def. 6).

(Def. 6)(i) For every $x$ holds $x \in \mathrm{dom}\,\blacksquare\mathrm{commute}(F)$ iff there exists a function $f$ such that $f \in \mathrm{dom}\,F$ and $x = \mathrm{commute}(f)$, and

(ii) for every function $f$ such that $f \in \mathrm{dom}\,\blacksquare\mathrm{commute}(F)$ holds $(\blacksquare\mathrm{commute}(F))(f) = F(\mathrm{commute}(f))$.

One can prove the following proposition

(8) For every function $F$ such that $\mathrm{dom}\,F = \{\emptyset\}$ holds $\blacksquare\mathrm{commute}(F) = F$.

Let $f$ be a function yielding function. Then $\mathrm{Frege}(f)$ is a many sorted function indexed by $\prod(\mathrm{dom}_\kappa f(\kappa))$ and it can be characterized by the condition:

(Def. 8)[2] For every function $g$ such that $g \in \prod(\mathrm{dom}_\kappa f(\kappa))$ holds $(\mathrm{Frege}(f))(g) = f \hookleftarrow g$.

Let us consider $I$, $A$, $B$. The functor $[\![A, B]\!]$ yielding a many sorted set indexed by $I$ is defined by:

(Def. 9) For every $i$ such that $i \in I$ holds $[\![A, B]\!](i) = [\!: A(i), B(i) :\!]$.

Let us consider $I$ and let $A$, $B$ be non-empty many sorted sets indexed by $I$. One can check that $[\![A, B]\!]$ is non-empty.

We now state the proposition

(9) Let $I$ be a non empty set, $J$ be a set, $A$, $B$ be many sorted sets indexed by $I$, and $f$ be a function from $J$ into $I$. Then $[\![A, B]\!] \cdot f = [\![A \cdot f, B \cdot f]\!]$.

[1] The definitions (Def. 3) and (Def. 4) have been removed.

[2] The definition (Def. 7) has been removed.

Let $I$ be a non empty set, let us consider $J$, let $A$, $B$ be non-empty many sorted sets indexed by $I$, let $p$ be a function from $J$ into $I^*$, let $r$ be a function from $J$ into $I$, let $j$ be a set, let $f$ be a function from $(A^\# \cdot p)(j)$ into $(A \cdot r)(j)$, and let $g$ be a function from $(B^\# \cdot p)(j)$ into $(B \cdot r)(j)$. Let us assume that $j \in J$. The functor $\rceil\rceil f, g\lceil\lceil$ yielding a function from $([\![A,B]\!]^\# \cdot p)(j)$ into $([\![A,B]\!] \cdot r)(j)$ is defined by:

(Def. 10) For every function $h$ such that $h \in ([\![A,B]\!]^\# \cdot p)(j)$ holds $\rceil\rceil f, g\lceil\lceil(h) = \langle f(\mathrm{pr1}(h)), g(\mathrm{pr2}(h))\rangle$.

Let $I$ be a non empty set, let us consider $J$, let $A$, $B$ be non-empty many sorted sets indexed by $I$, let $p$ be a function from $J$ into $I^*$, let $r$ be a function from $J$ into $I$, let $F$ be a many sorted function from $A^\# \cdot p$ into $A \cdot r$, and let $G$ be a many sorted function from $B^\# \cdot p$ into $B \cdot r$. The functor $\rceil\rceil F, G\lceil\lceil$ yields a many sorted function from $[\![A,B]\!]^\# \cdot p$ into $[\![A,B]\!] \cdot r$ and is defined by the condition (Def. 11).

(Def. 11) Let given $j$. Suppose $j \in J$. Let $f$ be a function from $(A^\# \cdot p)(j)$ into $(A \cdot r)(j)$ and $g$ be a function from $(B^\# \cdot p)(j)$ into $(B \cdot r)(j)$. If $f = F(j)$ and $g = G(j)$, then $\rceil\rceil F, G\lceil\lceil(j) = \rceil\rceil f, g\lceil\lceil$.

## 3. Family of Many Sorted Universal Algebras

Let us consider $I$, $S$. A many sorted set indexed by $I$ is said to be an algebra family of $I$ over $S$ if:

(Def. 12) For every $i$ such that $i \in I$ holds $\mathrm{it}(i)$ is a non-empty algebra over $S$.

Let $I$ be a non empty set, let us consider $S$, let $A$ be an algebra family of $I$ over $S$, and let $i$ be an element of $I$. Then $A(i)$ is a non-empty algebra over $S$.

Let $S$ be a non empty many sorted signature and let $U_1$ be an algebra over $S$. The functor $|U_1|$ yielding a set is defined as follows:

(Def. 13) $|U_1| = \bigcup \mathrm{rng}(\text{the sorts of } U_1)$.

Let $S$ be a non empty many sorted signature and let $U_1$ be a non-empty algebra over $S$. Note that $|U_1|$ is non empty.

Let $I$ be a non empty set, let $S$ be a non empty many sorted signature, and let $A$ be an algebra family of $I$ over $S$. The functor $|A|$ yields a set and is defined by:

(Def. 14) $|A| = \bigcup\{|A(i)| : i \text{ ranges over elements of } I\}$.

Let $I$ be a non empty set, let $S$ be a non empty many sorted signature, and let $A$ be an algebra family of $I$ over $S$. Note that $|A|$ is non empty.

## 4. Product of Many Sorted Universal Algebras

Next we state two propositions:

(10) Let $S$ be a non void non empty many sorted signature, $U_0$ be an algebra over $S$, and $o$ be an operation symbol of $S$. Then $\mathrm{Args}(o, U_0) = \prod((\text{the sorts of } U_0) \cdot \mathrm{Arity}(o))$ and $\mathrm{dom}((\text{the sorts of } U_0) \cdot \mathrm{Arity}(o)) = \mathrm{dom}\,\mathrm{Arity}(o)$ and $\mathrm{Result}(o, U_0) = (\text{the sorts of } U_0)(\text{the result sort of } o)$.

(11) Let $S$ be a non void non empty many sorted signature, $U_0$ be an algebra over $S$, and $o$ be an operation symbol of $S$. If $\mathrm{Arity}(o) = \emptyset$, then $\mathrm{Args}(o, U_0) = \{\emptyset\}$.

Let us consider $S$ and let $U_1$, $U_2$ be non-empty algebras over $S$. The functor $[\!: U_1, U_2 :\!]$ yielding an algebra over $S$ is defined by:

(Def. 15) $[\!: U_1, U_2 :\!] = \langle [\![\text{the sorts of } U_1, \text{the sorts of } U_2]\!], \rceil\rceil\text{the characteristics of } U_1, \text{the characteristics of } U_2\lceil\lceil\rangle$.

Let us consider $S$ and let $U_1$, $U_2$ be non-empty algebras over $S$. Note that $[\!: U_1, U_2 :\!]$ is strict.

Let us consider $I$, $S$, let $s$ be a sort symbol of $S$, and let $A$ be an algebra family of $I$ over $S$. The functor $\mathrm{Carrier}(A, s)$ yields a many sorted set indexed by $I$ and is defined as follows:

(Def. 16)(i) For every set $i$ such that $i \in I$ there exists an algebra $U_0$ over $S$ such that $U_0 = A(i)$ and $(\mathrm{Carrier}(A,s))(i) =$ (the sorts of $U_0)(s)$ if $I \neq \emptyset$,

(ii) $\mathrm{Carrier}(A,s) = \emptyset$, otherwise.

Let us consider $I$, $S$, let $s$ be a sort symbol of $S$, and let $A$ be an algebra family of $I$ over $S$. Observe that $\mathrm{Carrier}(A,s)$ is non-empty.

Let us consider $I$, $S$ and let $A$ be an algebra family of $I$ over $S$. The functor $\mathrm{SORTS}(A)$ yields a many sorted set indexed by the carrier of $S$ and is defined as follows:

(Def. 17) For every sort symbol $s$ of $S$ holds $(\mathrm{SORTS}(A))(s) = \prod \mathrm{Carrier}(A,s)$.

Let us consider $I$, $S$ and let $A$ be an algebra family of $I$ over $S$. Observe that $\mathrm{SORTS}(A)$ is non-empty.

Let us consider $I$, let $S$ be a non empty many sorted signature, and let $A$ be an algebra family of $I$ over $S$. The functor $\mathrm{OPER}(A)$ yields a many sorted function indexed by $I$ and is defined by:

(Def. 18)(i) For every set $i$ such that $i \in I$ there exists an algebra $U_0$ over $S$ such that $U_0 = A(i)$ and $(\mathrm{OPER}(A))(i) =$ the characteristics of $U_0$ if $I \neq \emptyset$,

(ii) $\mathrm{OPER}(A) = \emptyset$, otherwise.

The following two propositions are true:

(12) Let $S$ be a non empty many sorted signature and $A$ be an algebra family of $I$ over $S$. Then $\mathrm{dom}\,\mathrm{uncurry}\,\mathrm{OPER}(A) = [:I,$ the operation symbols of $S:]$.

(13) Let $I$ be a non empty set, $S$ be a non void non empty many sorted signature, $A$ be an algebra family of $I$ over $S$, and $o$ be an operation symbol of $S$. Then $\mathrm{commute}(\mathrm{OPER}(A)) \in ((\mathrm{rng}\,\mathrm{uncurry}\,\mathrm{OPER}(A))^I)^{\text{the operation symbols of } S}$.

Let $I$ be a set, let $S$ be a non void non empty many sorted signature, let $A$ be an algebra family of $I$ over $S$, and let $o$ be an operation symbol of $S$. The functor $A(o)$ yielding a many sorted function indexed by $I$ is defined by:

(Def. 19) $A(o) = (\mathrm{commute}(\mathrm{OPER}(A)))(o)$.

One can prove the following propositions:

(14) Let $I$ be a non empty set, $i$ be an element of $I$, $S$ be a non void non empty many sorted signature, $A$ be an algebra family of $I$ over $S$, and $o$ be an operation symbol of $S$. Then $A(o)(i) = \mathrm{Den}(o, A(i))$.

(15) Let $I$ be a non empty set, $S$ be a non void non empty many sorted signature, $A$ be an algebra family of $I$ over $S$, $o$ be an operation symbol of $S$, and $x$ be a set. If $x \in \mathrm{rng}\,\mathrm{Frege}(A(o))$, then $x$ is a function.

(16) Let $I$ be a non empty set, $S$ be a non void non empty many sorted signature, $A$ be an algebra family of $I$ over $S$, $o$ be an operation symbol of $S$, and $f$ be a function. If $f \in \mathrm{rng}\,\mathrm{Frege}(A(o))$, then $\mathrm{dom}\, f = I$ and for every element $i$ of $I$ holds $f(i) \in \mathrm{Result}(o, A(i))$.

(17) Let $I$ be a non empty set, $S$ be a non void non empty many sorted signature, $A$ be an algebra family of $I$ over $S$, $o$ be an operation symbol of $S$, and $f$ be a function. Suppose $f \in \mathrm{dom}\,\mathrm{Frege}(A(o))$. Then $\mathrm{dom}\, f = I$ and for every element $i$ of $I$ holds $f(i) \in \mathrm{Args}(o, A(i))$ and $\mathrm{rng}\, f \subseteq |A|^{\mathrm{dom}\,\mathrm{Arity}(o)}$.

(18) Let $I$ be a non empty set, $S$ be a non void non empty many sorted signature, $A$ be an algebra family of $I$ over $S$, and $o$ be an operation symbol of $S$. Then $\mathrm{dom}(\mathrm{dom}_\kappa A(o)(\kappa)) = I$ and for every element $i$ of $I$ holds $(\mathrm{dom}_\kappa A(o)(\kappa))(i) = \mathrm{Args}(o, A(i))$.

Let us consider $I$, let $S$ be a non void non empty many sorted signature, and let $A$ be an algebra family of $I$ over $S$. The functor $\mathrm{OPS}(A)$ yields a many sorted function from $(\mathrm{SORTS}(A))^\# \cdot$ the arity of $S$ into $\mathrm{SORTS}(A) \cdot$ the result sort of $S$ and is defined as follows:

(Def. 20)(i) For every operation symbol $o$ of $S$ holds $(\mathrm{OPS}(A))(o) = (\mathrm{Arity}(o) = \emptyset \to \mathrm{commute}(A(o)), \blacksquare\mathrm{commute}(\mathrm{Frege}(A(o))))$ if $I \neq \emptyset$,

(ii) TRUE, otherwise.

Let $I$ be a set, let $S$ be a non void non empty many sorted signature, and let $A$ be an algebra family of $I$ over $S$. The functor $\prod A$ yielding an algebra over $S$ is defined by:

(Def. 21) $\prod A = \langle \mathrm{SORTS}(A), \mathrm{OPS}(A) \rangle$.

Let $I$ be a set, let $S$ be a non void non empty many sorted signature, and let $A$ be an algebra family of $I$ over $S$. Note that $\prod A$ is strict.

One can prove the following proposition

(20)[3] Let $S$ be a non void non empty many sorted signature and $A$ be an algebra family of $I$ over $S$. Then the sorts of $\prod A = \mathrm{SORTS}(A)$ and the characteristics of $\prod A = \mathrm{OPS}(A)$.

## REFERENCES


[1] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_5.html.

[2] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[3] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/funct_6.html.

[4] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/dtconstr.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[8] Czesław Byliński. A classical first order language. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/cqc_lang.html.

[9] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[10] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[12] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[13] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[14] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[15] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.


[3] The proposition (19) has been removed.

[17] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

*Received April 25, 1994*

*Published January 2, 2004*