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Paracompact and Metrizable Spaces

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Summary. We give an example of a compact space. Next we define a locally finite subset family of a topological space and a paracompact topological space. An open sets family of a metric space we define next and it has been shown that the metric space with any open sets family is a topological space. Next we define metrizable space.

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The articles [12], [4], [14], [1], [13], [8], [6], [10], [7], [5], [11], [15], [2], [3], and [9] provide the notation and terminology for this paper.

We adopt the following convention: P_1 denotes a metric structure, x denotes an element of P_1, and r, p, q denote real numbers.

We now state the proposition

(1) If $r \leq p$, then $\mathrm{Ball}(x,r) \subseteq \mathrm{Ball}(x,p)$.

In the sequel T denotes a topological space and A denotes a subset of T.

The following three propositions are true:

(2) $\overline{A} \neq \emptyset$ iff $A \neq \emptyset$.

(3) If $\overline{A} = \emptyset$, then $A = \emptyset$.

(4) $\overline{A}$ is closed.

For simplicity, we adopt the following convention: T is a non empty topological space, x is a point of T, V, W are subsets of T, and F_1 is a family of subsets of T.

Next we state the proposition

(5) If F_1 is a cover of T, then for every x there exists W such that $x \in W$ and $W \in F_1$.

Let X be a set. Then 2^X is a non empty family of subsets of X.

Let D be a set. The functor $\{D\}_{\mathrm{top}}$ yielding a topological structure is defined as follows:

(Def. 1) $\{D\}_{\mathrm{top}} = \langle D, 2^D \rangle$.

Let D be a set. One can check that $\{D\}_{\mathrm{top}}$ is strict and topological space-like.

Let D be a non empty set. Note that $\{D\}_{\mathrm{top}}$ is non empty.

In the sequel a is a set.

One can prove the following propositions:

(7)[1] The topology of $\{a\}_{\rm top} = 2^a$.

(8) The carrier of $\{a\}_{\rm top} = a$.

(9) $\{\{a\}\}_{\rm top}$ is compact.

(10) If T is a T_2 space, then $\{x\}$ is closed.

We adopt the following rules: x is a point of T, A, B are subsets of T, and F_1, G_1 are families of subsets of T.

Let T be a topological structure and let I_1 be a family of subsets of T. We say that I_1 is locally finite if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x be a point of T. Then there exists a subset W of T such that $x \in W$ and W is open and $\{V; V$ ranges over subsets of T: $V \in I_1 \wedge V$ meets $W\}$ is finite.

The following propositions are true:

(11) For every W holds $\{V : V \in F_1 \wedge V$ meets $W\} \subseteq F_1$.

(12) If $F_1 \subseteq G_1$ and G_1 is locally finite, then F_1 is locally finite.

(13) If F_1 is finite, then F_1 is locally finite.

Let T be a topological structure and let F_1 be a family of subsets of T. The functor $\mathrm{clf}\,F_1$ yielding a family of subsets of T is defined as follows:

(Def. 3) For every subset Z of T holds $Z \in \mathrm{clf}\,F_1$ iff there exists a subset W of T such that $Z = \overline{W}$ and $W \in F_1$.

One can prove the following propositions:

(14) $\mathrm{clf}\,F_1$ is closed.

(15) If $F_1 = \emptyset$, then $\mathrm{clf}\,F_1 = \emptyset$.

(16) If $F_1 = \{V\}$, then $\mathrm{clf}\,F_1 = \{\overline{V}\}$.

(17) If $F_1 \subseteq G_1$, then $\mathrm{clf}\,F_1 \subseteq \mathrm{clf}\,G_1$.

(18) $\mathrm{clf}(F_1 \cup G_1) = \mathrm{clf}\,F_1 \cup \mathrm{clf}\,G_1$.

(19) If F_1 is finite, then $\overline{\bigcup F_1} = \bigcup \mathrm{clf}\,F_1$.

(20) F_1 is finer than $\mathrm{clf}\,F_1$.

The scheme *Lambda1top* deals with a topological space $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, a family $\mathcal{C}$ of subsets of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

There exists a function f from $\mathcal{B}$ into $\mathcal{C}$ such that for every subset Z of $\mathcal{A}$ such that $Z \in \mathcal{B}$ holds $f(Z) = \mathcal{F}(Z)$

provided the parameters meet the following condition:

- For every subset Z of $\mathcal{A}$ such that $Z \in \mathcal{B}$ holds $\mathcal{F}(Z) \in \mathcal{C}$.

The following propositions are true:

(21) If F_1 is locally finite, then $\mathrm{clf}\,F_1$ is locally finite.

(22) $\bigcup F_1 \subseteq \bigcup \mathrm{clf}\,F_1$.

(23) If F_1 is locally finite, then $\overline{\bigcup F_1} = \bigcup \mathrm{clf}\,F_1$.

(24) If F_1 is locally finite and closed, then $\bigcup F_1$ is closed.

[1] The proposition (6) has been removed.

Let I_1 be a topological structure. We say that I_1 is paracompact if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let F_1 be a family of subsets of I_1. Suppose F_1 is a cover of I_1 and open. Then there exists a family G_1 of subsets of I_1 which is open, a cover of I_1, finer than F_1, and locally finite.

Let us observe that there exists a non empty topological space which is paracompact.

We now state four propositions:

(25) If T is compact, then T is paracompact.

(26) Suppose that

(i) T is paracompact,

(ii) A is closed,

(iii) B is closed,

(iv) A misses B, and

(v) for every x such that $x \in B$ there exist subsets V, W of T such that V is open and W is open and $A \subseteq V$ and $x \in W$ and V misses W.

Then there exist subsets Y, Z of T such that Y is open and Z is open and $A \subseteq Y$ and $B \subseteq Z$ and Y misses Z.

(27) If T is a T_2 space and paracompact, then T is a T_3 space.

(28) If T is a T_2 space and paracompact, then T is a T_4 space.

We use the following convention: x, y, z denote elements of P_1 and V, W denote subsets of P_1.

The scheme *SubFamExM* deals with a metric structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let us consider P_1. The open set family of P_1 yields a family of subsets of P_1 and is defined as follows:

(Def. 5) For every V holds $V \in$ the open set family of P_1 iff for every x such that $x \in V$ there exists r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq V$.

The following propositions are true:

(29) For every x there exists r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq$ the carrier of P_1.

(30) For every real number r such that P_1 is triangle and $y \in \mathrm{Ball}(x, r)$ there exists p such that $p > 0$ and $\mathrm{Ball}(y, p) \subseteq \mathrm{Ball}(x, r)$.

(31) If P_1 is triangle and $y \in \mathrm{Ball}(x, r) \cap \mathrm{Ball}(z, p)$, then there exists q such that $\mathrm{Ball}(y, q) \subseteq \mathrm{Ball}(x, r)$ and $\mathrm{Ball}(y, q) \subseteq \mathrm{Ball}(z, p)$.

(33)[2] For every real number r such that P_1 is triangle holds $\mathrm{Ball}(x, r) \in$ the open set family of P_1.

(34) The carrier of $P_1 \in$ the open set family of P_1.

(35) Let given V, W. Suppose $V \in$ the open set family of P_1 and $W \in$ the open set family of P_1. Then $V \cap W \in$ the open set family of P_1.

(36) Let A be a family of subsets of P_1. Suppose $A \subseteq$ the open set family of P_1. Then $\bigcup A \in$ the open set family of P_1.

[2] The proposition (32) has been removed.

(37) $\langle$the carrier of P_1, the open set family of $P_1\rangle$ is a topological space.

Let us consider P_1. The functor $(P_1)_{\rm top}$ yielding a topological structure is defined as follows:

(Def. 6) $(P_1)_{\rm top} = \langle$the carrier of P_1, the open set family of $P_1\rangle$.

Let us consider P_1. Observe that $(P_1)_{\rm top}$ is strict and topological space-like.
Let P_1 be a non empty metric structure. One can check that $(P_1)_{\rm top}$ is non empty.
We now state the proposition

(38) For every non empty metric space P_1 holds $(P_1)_{\rm top}$ is a T_2 space.

Let D be a set and let f be a function from $[:D, D:]$ into $\mathbb{R}$. We say that f is a metric of D if and only if:

(Def. 7) For all elements a, b, c of D holds $f(a, b) = 0$ iff $a = b$ and $f(a, b) = f(b, a)$ and $f(a, c) \leq f(a, b) + f(b, c)$.

The following proposition is true

(39) Let D be a set and f be a function from $[:D, D:]$ into $\mathbb{R}$. Then f is a metric of D if and only if $\langle D, f\rangle$ is a metric space.

Let D be a non empty set and let f be a function from $[:D, D:]$ into $\mathbb{R}$. Let us assume that f is a metric of D. The functor $\mathrm{MetrSp}(D, f)$ yields a strict non empty metric space and is defined by:

(Def. 8) $\mathrm{MetrSp}(D, f) = \langle D, f\rangle$.

Let I_1 be a non empty topological structure. We say that I_1 is metrizable if and only if the condition (Def. 9) is satisfied.

(Def. 9) There exists a function f from $[:$the carrier of I_1, the carrier of $I_1:]$ into $\mathbb{R}$ such that f is a metric of the carrier of I_1 and the open set family of MetrSp((the carrier of I_1), f) = the topology of I_1.

Let us note that there exists a non empty topological space which is strict and metrizable.

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