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Partial Functions from a Domain to a Domain

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Summary. The value of a partial function from a domain to a domain and a inverse partial function are introduced. The value and inverse function were defined in the article [1], but new definitions are introduced. The basic properties of the value, the inverse partial function, the identity partial function, the composition of partial functions, the $1-1$ partial function, the restriction of a partial function, the image, the inverse image and the graph are proved. Constant partial functions are introduced, too.

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The articles [5], [7], [8], [9], [1], [2], [4], [3], and [6] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: x, y, X, Y are sets, C, D, E are non empty sets, S_1 is a subset of C, S_2 is a subset of D, S_3 is a subset of E, c, c_1, c_2 are elements of C, d is an element of D, e is an element of E, f, f_1, g are partial functions from C to D, t is a partial function from D to C, s is a partial function from D to E, h is a partial function from C to E, and F is a partial function from D to D.

We now state several propositions:

(3)[1] If $\operatorname{dom} f = \operatorname{dom} g$ and for every c such that $c \in \operatorname{dom} f$ holds $f_c = g_c$, then $f = g$.

(4) $y \in \operatorname{rng} f$ iff there exists c such that $c \in \operatorname{dom} f$ and $y = f_c$.

(6)[2] $h = s \cdot f$ if and only if the following conditions are satisfied:

(i) for every c holds $c \in \operatorname{dom} h$ iff $c \in \operatorname{dom} f$ and $f_c \in \operatorname{dom} s$, and

(ii) for every c such that $c \in \operatorname{dom} h$ holds $h_c = s_{f_c}$.

(9)[3] If $c \in \operatorname{dom} f$ and $f_c \in \operatorname{dom} s$, then $(s \cdot f)_c = s_{f_c}$.

(10) If $\operatorname{rng} f \subseteq \operatorname{dom} s$ and $c \in \operatorname{dom} f$, then $(s \cdot f)_c = s_{f_c}$.

Let us consider D and let us consider S_2. Then $\mathrm{id}_{(S_2)}$ is a partial function from D to D.

We now state several propositions:

(12)[4] $F = \mathrm{id}_{(S_2)}$ iff $\operatorname{dom} F = S_2$ and for every d such that $d \in S_2$ holds $F_d = d$.

(14)[5] If $d \in \operatorname{dom} F \cap S_2$, then $F_d = (F \cdot \mathrm{id}_{(S_2)})_d$.

[1] The propositions (1) and (2) have been removed.
[2] The proposition (5) has been removed.
[3] The propositions (7) and (8) have been removed.
[4] The proposition (11) has been removed.
[5] The proposition (13) has been removed.

(15) $d \in \operatorname{dom}(\operatorname{id}_{(S_2)} \cdot F)$ iff $d \in \operatorname{dom} F$ and $F_d \in S_2$.

(16) If for all c_1, c_2 such that $c_1 \in \operatorname{dom} f$ and $c_2 \in \operatorname{dom} f$ and $f_{c_1} = f_{c_2}$ holds $c_1 = c_2$, then f is one-to-one.

(17) If f is one-to-one and $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ and $f_x = f_y$, then $x = y$.

Let us mention that $\emptyset$ is one-to-one.

Let us consider X, Y. Observe that there exists a partial function from X to Y which is one-to-one.

Let us consider X, Y and let f be an one-to-one partial function from X to Y. Then f^{-1} is a partial function from Y to X.

One can prove the following propositions:

(18) Let f be an one-to-one partial function from C to D and g be a partial function from D to C. Then $g = f^{-1}$ if and only if the following conditions are satisfied:

(i) $\operatorname{dom} g = \operatorname{rng} f$, and

(ii) for all d, c holds $d \in \operatorname{rng} f$ and $c = g_d$ iff $c \in \operatorname{dom} f$ and $d = f_c$.

(22)[6] For every one-to-one partial function f from C to D such that $c \in \operatorname{dom} f$ holds $c = (f^{-1})_{f_c}$ and $c = (f^{-1} \cdot f)_c$.

(23) For every one-to-one partial function f from C to D such that $d \in \operatorname{rng} f$ holds $d = f_{(f^{-1})_d}$ and $d = (f \cdot f^{-1})_d$.

(24) Suppose f is one-to-one and $\operatorname{dom} f = \operatorname{rng} t$ and $\operatorname{rng} f = \operatorname{dom} t$ and for all c, d such that $c \in \operatorname{dom} f$ and $d \in \operatorname{dom} t$ holds $f_c = d$ iff $t_d = c$. Then $t = f^{-1}$.

(32)[7] $g = f \restriction X$ iff $\operatorname{dom} g = \operatorname{dom} f \cap X$ and for every c such that $c \in \operatorname{dom} g$ holds $g_c = f_c$.

(34)[8] If $c \in \operatorname{dom} f \cap X$, then $(f \restriction X)_c = f_c$.

(35) If $c \in \operatorname{dom} f$ and $c \in X$, then $(f \restriction X)_c = f_c$.

(36) If $c \in \operatorname{dom} f$ and $c \in X$, then $f_c \in \operatorname{rng}(f \restriction X)$.

Let us consider C, D and let us consider X, f. Then $X \restriction f$ is a partial function from C to D.

Next we state a number of propositions:

(37) $g = X \restriction f$ if and only if the following conditions are satisfied:

(i) for every c holds $c \in \operatorname{dom} g$ iff $c \in \operatorname{dom} f$ and $f_c \in X$, and

(ii) for every c such that $c \in \operatorname{dom} g$ holds $g_c = f_c$.

(38) $c \in \operatorname{dom}(X \restriction f)$ iff $c \in \operatorname{dom} f$ and $f_c \in X$.

(39) If $c \in \operatorname{dom}(X \restriction f)$, then $(X \restriction f)_c = f_c$.

(40) $S_2 = f^\circ X$ iff for every d holds $d \in S_2$ iff there exists c such that $c \in \operatorname{dom} f$ and $c \in X$ and $d = f_c$.

(41) $d \in (f$ **qua** relation between C and $D)^\circ X$ iff there exists c such that $c \in \operatorname{dom} f$ and $c \in X$ and $d = f_c$.

(42) If $c \in \operatorname{dom} f$, then $f^\circ\{c\} = \{f_c\}$.

(43) If $c_1 \in \operatorname{dom} f$ and $c_2 \in \operatorname{dom} f$, then $f^\circ\{c_1, c_2\} = \{f_{c_1}, f_{c_2}\}$.

(44) $S_1 = f^{-1}(X)$ iff for every c holds $c \in S_1$ iff $c \in \operatorname{dom} f$ and $f_c \in X$.

[6] The propositions (19)–(21) have been removed.
[7] The propositions (25)–(31) have been removed.
[8] The proposition (33) has been removed.

(46)[9] For every f there exists a function g from C into D such that for every c such that $c \in \operatorname{dom} f$ holds $g(c) = f_c$.

(47) $f \approx g$ iff for every c such that $c \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f_c = g_c$.

In this article we present several logical schemes. The scheme *PartFuncExD* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that
(i) for every element d of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ iff there exists an element c of $\mathcal{B}$ such that $\mathcal{P}[d, c]$, and
(ii) for every element d of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $\mathcal{P}[d, f_d]$

for all values of the parameters.

The scheme *LambdaPFD* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every element d of $\mathcal{A}$ holds $d \in \operatorname{dom} f$ iff $\mathcal{P}[d]$ and for every element d of $\mathcal{A}$ such that $d \in \operatorname{dom} f$ holds $f_d = \mathcal{F}(d)$

for all values of the parameters.

The scheme *UnPartFuncD* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

Let f, g be partial functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose that
(i) $\operatorname{dom} f = \mathcal{C}$,
(ii) for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} f$ holds $f_c = \mathcal{F}(c)$,
(iii) $\operatorname{dom} g = \mathcal{C}$, and
(iv) for every element c of $\mathcal{A}$ such that $c \in \operatorname{dom} g$ holds $g_c = \mathcal{F}(c)$.
Then $f = g$

for all values of the parameters.

Let us consider C, D and let us consider S_1, d. Then $S_1 \longmapsto d$ is a partial function from C to D.

One can prove the following propositions:

(48) If $c \in S_1$, then $(S_1 \longmapsto d)_c = d$.

(49) If for every c such that $c \in \operatorname{dom} f$ holds $f_c = d$, then $f = \operatorname{dom} f \longmapsto d$.

(50) If $c \in \operatorname{dom} f$, then $f \cdot (S_3 \longmapsto c) = S_3 \longmapsto f_c$.

(51) $\mathrm{id}_{(S_1)}$ is total iff $S_1 = C$.

(52) If $S_1 \longmapsto d$ is total, then $S_1 \neq \emptyset$.

(53) $S_1 \longmapsto d$ is total iff $S_1 = C$.

Let us consider C, D and let us consider f, X. We say that f is a constant on X if and only if:

(Def. 3)[10] There exists d such that for every c such that $c \in X \cap \operatorname{dom} f$ holds $f_c = d$.

Next we state a number of propositions:

(55)[11] f is a constant on X iff for all c_1, c_2 such that $c_1 \in X \cap \operatorname{dom} f$ and $c_2 \in X \cap \operatorname{dom} f$ holds $f_{c_1} = f_{c_2}$.

(56) If X meets $\operatorname{dom} f$, then f is a constant on X iff there exists d such that $\operatorname{rng}(f{\restriction}X) = \{d\}$.

(57) If f is a constant on X and $Y \subseteq X$, then f is a constant on Y.

(58) If X misses $\operatorname{dom} f$, then f is a constant on X.

(59) If $f{\restriction}S_1 = \operatorname{dom}(f{\restriction}S_1) \longmapsto d$, then f is a constant on S_1.

[9] The proposition (45) has been removed.
[10] The definitions (Def. 1) and (Def. 2) have been removed.
[11] The proposition (54) has been removed.

(60) f is a constant on $\{x\}$.

(61) If f is a constant on X and a constant on Y and $X \cap Y$ meets $\mathrm{dom} f$, then f is a constant on $X \cup Y$.

(62) If f is a constant on Y, then $f \restriction X$ is a constant on Y.

(63) $S_1 \longmapsto d$ is a constant on S_1.

(64) $f \subseteq g$ iff $\mathrm{dom} f \subseteq \mathrm{dom} g$ and for every c such that $c \in \mathrm{dom} f$ holds $f_c = g_c$.

(65) $c \in \mathrm{dom} f$ and $d = f_c$ iff $\langle c, d \rangle \in f$.

(66) If $\langle c, e \rangle \in s \cdot f$, then $\langle c, f_c \rangle \in f$ and $\langle f_c, e \rangle \in s$.

(67) If $f = \{\langle c, d \rangle\}$, then $f_c = d$.

(68) If $\mathrm{dom} f = \{c\}$, then $f = \{\langle c, f_c \rangle\}$.

(69) If $f_1 = f \cap g$ and $c \in \mathrm{dom} f_1$, then $(f_1)_c = f_c$ and $(f_1)_c = g_c$.

(70) If $c \in \mathrm{dom} f$ and $f_1 = f \cup g$, then $(f_1)_c = f_c$.

(71) If $c \in \mathrm{dom} g$ and $f_1 = f \cup g$, then $(f_1)_c = g_c$.

(72) If $c \in \mathrm{dom} f_1$ and $f_1 = f \cup g$, then $(f_1)_c = f_c$ or $(f_1)_c = g_c$.

(73) $c \in \mathrm{dom} f$ and $c \in S_1$ iff $\langle c, f_c \rangle \in f \restriction S_1$.

(74) $c \in \mathrm{dom} f$ and $f_c \in S_2$ iff $\langle c, f_c \rangle \in S_2 \restriction f$.

(75) $c \in f^{-1}(S_2)$ iff $\langle c, f_c \rangle \in f$ and $f_c \in S_2$.

(76) f is a constant on X iff there exists d such that for every c such that $c \in X \cap \mathrm{dom} f$ holds $f(c) = d$.

(77) f is a constant on X iff for all c_1, c_2 such that $c_1 \in X \cap \mathrm{dom} f$ and $c_2 \in X \cap \mathrm{dom} f$ holds $f(c_1) = f(c_2)$.

(78) If $d \in f^{\circ} X$, then there exists c such that $c \in \mathrm{dom} f$ and $c \in X$ and $d = f(c)$.

(79) If f is one-to-one, then $d \in \mathrm{rng} f$ and $c = f^{-1}(d)$ iff $c \in \mathrm{dom} f$ and $d = f(c)$.

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