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Parallelity Spaces[1]

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Summary. In the monography [6] W. Szmielew introduced the parallelity planes $\langle S; \parallel\rangle$, where $\parallel \subseteq S \times S \times S \times S$. In this text we omit upper bound axiom which must be satisfied by the parallelity planes (see also E.Kusak [4]). Further we will list those theorems which remain true when we pass from the parallelity planes to the parallelity spaces. We construct a model of the parallelity space in Abelian group $\langle F \times F \times F; +_F, -_F, \mathbf{0}_F\rangle$, where F is a field.

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The articles [8], [3], [11], [9], [7], [2], [1], [10], and [5] provide the notation and terminology for this paper.

We adopt the following convention: F denotes a field, a, b, c, f, g, h denote elements of F, and x, y denote elements of [: the carrier of F, the carrier of F, the carrier of F :].

Let us consider F. The functor $+_F$ yielding a binary operation on [: the carrier of F, the carrier of F, the carrier of F :] is defined by:

(Def. 1) $+_F(x, y) = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

Let us consider F, x, y. The functor $x + y$ yielding an element of [: the carrier of F, the carrier of F, the carrier of F :] is defined as follows:

(Def. 2) $x + y = +_F(x, y)$.

The following two propositions are true:

(3)[1] $x + y = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

(4) $\langle a, b, c\rangle + \langle f, g, h\rangle = \langle a + f, b + g, c + h\rangle$.

Let us consider F. The functor $-_F$ yields a unary operation on [: the carrier of F, the carrier of F, the carrier of F :] and is defined as follows:

(Def. 3) $-_F(x) = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}}\rangle$.

Let us consider F, x. The functor $-x$ yields an element of [: the carrier of F, the carrier of F, the carrier of F :] and is defined by:

(Def. 4) $-x = -_F(x)$.

The following proposition is true

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[1] The propositions (1) and (2) have been removed.

(7)[2] $-x = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

In the sequel S denotes a set.

Let us consider S. A set is called a 4-ary relation over S if:

(Def. 5) It $\subseteq [\!: S, S, S, S :\!]$.

We introduce parallelity structures which are extensions of 1-sorted structure and are systems
⟨ a carrier, a parallelity ⟩,
where the carrier is a set and the parallelity is a 4-ary relation over the carrier.

Let us observe that there exists a parallelity structure which is non empty.

In the sequel F is a field and P_1 is a non empty parallelity structure.

Let us consider P_1 and let a, b, c, d be elements of P_1. The predicate $a, b \parallel c, d$ is defined as follows:

(Def. 6) $\langle a, b, c, d \rangle \in$ the parallelity of P_1.

Let us consider F. The functor $F^{\mathbf{3}}$ yields a set and is defined by:

(Def. 7) $F^{\mathbf{3}} = [\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$.

Let us consider F. One can verify that $F^{\mathbf{3}}$ is non empty.

Let us consider F. The functor $(F^{\mathbf{3}})^{\mathbf{4}}$ yields a set and is defined by:

(Def. 8) $(F^{\mathbf{3}})^{\mathbf{4}} = [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

Let us consider F. Note that $(F^{\mathbf{3}})^{\mathbf{4}}$ is non empty.

We adopt the following convention: x is a set and a, b, c, d, e, f, g, h are elements of $[\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$.

Let us consider F. The functor $\mathbf{Par}'_F$ yielding a set is defined by the condition (Def. 9).

(Def. 9) $x \in \mathbf{Par}'_F$ if and only if the following conditions are satisfied:

(i) $x \in (F^{\mathbf{3}})^{\mathbf{4}}$, and

(ii) there exist a, b, c, d such that $x = \langle a, b, c, d \rangle$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{2}} - d_{\mathbf{2}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{2}} - b_{\mathbf{2}}) = 0_F$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$ and $(a_{\mathbf{2}} - b_{\mathbf{2}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{2}} - d_{\mathbf{2}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$.

Next we state the proposition

(13)[3] $\mathbf{Par}'_F \subseteq [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

Let us consider F. The functor $\mathbf{Par}_F$ yielding a 4-ary relation over $F^{\mathbf{3}}$ is defined as follows:

(Def. 10) $\mathbf{Par}_F = \mathbf{Par}'_F$.

Let us consider F. The functor Aff_{F^3} yielding a parallelity structure is defined by:

(Def. 11) $\mathrm{Aff}_{F^3} = \langle F^{\mathbf{3}}, \mathbf{Par}_F \rangle$.

Let us consider F. Observe that Aff_{F^3} is strict and non empty.

We now state two propositions:

(16)[4] The carrier of $\mathrm{Aff}_{F^3} = F^{\mathbf{3}}$.

(17) The parallelity of $\mathrm{Aff}_{F^3} = \mathbf{Par}_F$.

In the sequel a, b, c, d, p, q, r, s are elements of Aff_{F^3}.

One can prove the following propositions:

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[2] The propositions (5) and (6) have been removed.
[3] The propositions (8)–(12) have been removed.
[4] The propositions (14) and (15) have been removed.

(18) $a,b \parallel c,d$ iff $\langle a,b,c,d\rangle \in \mathbf{Par}_F$.

(19) $\langle a,b,c,d\rangle \in \mathbf{Par}_F$ if and only if the following conditions are satisfied:

(i) $\langle a,b,c,d\rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$, and

(ii) there exist e, f, g, h such that $\langle a,b,c,d\rangle = \langle e,f,g,h\rangle$ and $(e_{\mathbf{1}}-f_{\mathbf{1}})\cdot(g_{\mathbf{2}}-h_{\mathbf{2}})-(g_{\mathbf{1}}-h_{\mathbf{1}})\cdot(e_{\mathbf{2}}-f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}}-f_{\mathbf{1}})\cdot(g_{\mathbf{3}}-h_{\mathbf{3}})-(g_{\mathbf{1}}-h_{\mathbf{1}})\cdot(e_{\mathbf{3}}-f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}}-f_{\mathbf{2}})\cdot(g_{\mathbf{3}}-h_{\mathbf{3}})-(g_{\mathbf{2}}-h_{\mathbf{2}})\cdot(e_{\mathbf{3}}-f_{\mathbf{3}}) = 0_F$.

(20) $a,b \parallel c,d$ if and only if the following conditions are satisfied:

(i) $\langle a,b,c,d\rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$, and

(ii) there exist e, f, g, h such that $\langle a,b,c,d\rangle = \langle e,f,g,h\rangle$ and $(e_{\mathbf{1}}-f_{\mathbf{1}})\cdot(g_{\mathbf{2}}-h_{\mathbf{2}})-(g_{\mathbf{1}}-h_{\mathbf{1}})\cdot(e_{\mathbf{2}}-f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}}-f_{\mathbf{1}})\cdot(g_{\mathbf{3}}-h_{\mathbf{3}})-(g_{\mathbf{1}}-h_{\mathbf{1}})\cdot(e_{\mathbf{3}}-f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}}-f_{\mathbf{2}})\cdot(g_{\mathbf{3}}-h_{\mathbf{3}})-(g_{\mathbf{2}}-h_{\mathbf{2}})\cdot(e_{\mathbf{3}}-f_{\mathbf{3}}) = 0_F$.

(21) The carrier of $\mathrm{Aff}_{F^3} = [\!:$ the carrier of F, the carrier of F, the carrier of $F\,:\!]$.

(22) $\langle a,b,c,d\rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$.

(23) $a,b \parallel c,d$ if and only if there exist e, f, g, h such that $\langle a,b,c,d\rangle = \langle e,f,g,h\rangle$ and $(e_{\mathbf{1}}-f_{\mathbf{1}})\cdot(g_{\mathbf{2}}-h_{\mathbf{2}})-(g_{\mathbf{1}}-h_{\mathbf{1}})\cdot(e_{\mathbf{2}}-f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}}-f_{\mathbf{1}})\cdot(g_{\mathbf{3}}-h_{\mathbf{3}})-(g_{\mathbf{1}}-h_{\mathbf{1}})\cdot(e_{\mathbf{3}}-f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}}-f_{\mathbf{2}})\cdot(g_{\mathbf{3}}-h_{\mathbf{3}})-(g_{\mathbf{2}}-h_{\mathbf{2}})\cdot(e_{\mathbf{3}}-f_{\mathbf{3}}) = 0_F$.

(24) $a,b \parallel b,a$.

(25) $a,b \parallel c,c$.

(26) If $a,b \parallel p,q$ and $a,b \parallel r,s$, then $p,q \parallel r,s$ or $a = b$.

(27) If $a,b \parallel a,c$, then $b,a \parallel b,c$.

(28) There exists d such that $a,b \parallel c,d$ and $a,c \parallel b,d$.

Let I_1 be a non empty parallelity structure. We say that I_1 is parallelity space-like if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let a, b, c, d, p, q, r, s be elements of I_1. Then

(i) $a,b \parallel b,a$,

(ii) $a,b \parallel c,c$,

(iii) if $a,b \parallel p,q$ and $a,b \parallel r,s$, then $p,q \parallel r,s$ or $a = b$,

(iv) if $a,b \parallel a,c$, then $b,a \parallel b,c$, and

(v) there exists an element x of I_1 such that $a,b \parallel c,x$ and $a,c \parallel b,x$.

One can verify that there exists a non empty parallelity structure which is strict and parallelity space-like.

A parallelity space is a parallelity space-like non empty parallelity structure.

We adopt the following convention: P_1 is a parallelity space and a, b, c, d, p, q, r, s are elements of P_1.

We now state a number of propositions:

(35)[5] $a,b \parallel a,b$.

(36) If $a,b \parallel c,d$, then $c,d \parallel a,b$.

(37) $a,a \parallel b,c$.

(38) If $a,b \parallel c,d$, then $b,a \parallel c,d$.

[5] The propositions (29)–(34) have been removed.

(39) If $a,b \parallel c,d$, then $a,b \parallel d,c$.

(40) If $a,b \parallel c,d$, then $b,a \parallel c,d$ and $a,b \parallel d,c$ and $b,a \parallel d,c$ and $c,d \parallel a,b$ and $d,c \parallel a,b$ and $c,d \parallel b,a$ and $d,c \parallel b,a$.

(41) Suppose $a,b \parallel a,c$. Then $a,c \parallel a,b$ and $b,a \parallel a,c$ and $a,b \parallel c,a$ and $a,c \parallel b,a$ and $b,a \parallel c,a$ and $c,a \parallel a,b$ and $c,a \parallel b,a$ and $b,a \parallel b,c$ and $a,b \parallel b,c$ and $b,a \parallel c,b$ and $b,c \parallel b,a$ and $a,b \parallel c,b$ and $c,b \parallel b,a$ and $b,c \parallel a,b$ and $c,b \parallel a,b$ and $c,a \parallel c,b$ and $a,c \parallel c,b$ and $c,a \parallel b,c$ and $a,c \parallel b,c$ and $c,b \parallel c,a$ and $b,c \parallel c,a$ and $c,b \parallel a,c$ and $b,c \parallel a,c$.

(42) If $a=b$ or $c=d$ or $a=c$ and $b=d$ or $a=d$ and $b=c$, then $a,b \parallel c,d$.

(43) If $a \neq b$ and $p,q \parallel a,b$ and $a,b \parallel r,s$, then $p,q \parallel r,s$.

(44) If $a,b \not\parallel a,c$, then $a \neq b$ and $b \neq c$ and $c \neq a$.

(45) If $a,b \not\parallel c,d$, then $a \neq b$ and $c \neq d$.

(47)[6] Suppose $a,b \not\parallel a,c$. Then $a,c \not\parallel a,b$ and $b,a \not\parallel a,c$ and $a,b \not\parallel c,a$ and $a,c \not\parallel b,a$ and $b,a \not\parallel c,a$ and $c,a \not\parallel a,b$ and $c,a \not\parallel b,a$ and $b,a \not\parallel b,c$ and $a,b \not\parallel b,c$ and $b,a \not\parallel c,b$ and $b,c \not\parallel b,a$ and $b,a \not\parallel c,b$ and $c,b \not\parallel b,a$ and $b,c \not\parallel a,b$ and $c,b \not\parallel a,b$ and $c,a \not\parallel c,b$ and $a,c \not\parallel c,b$ and $c,a \not\parallel b,c$ and $a,c \not\parallel b,c$ and $c,b \not\parallel c,a$ and $b,c \not\parallel c,a$ and $c,b \not\parallel a,c$ and $b,c \not\parallel a,c$.

(48) If $a,b \not\parallel c,d$ and $a,b \parallel p,q$ and $c,d \parallel r,s$ and $p \neq q$ and $r \neq s$, then $p,q \not\parallel r,s$.

(49) If $a,b \not\parallel a,c$ and $a,b \parallel p,q$ and $a,c \parallel p,r$ and $b,c \parallel q,r$ and $p \neq q$, then $p,q \not\parallel p,r$.

(50) If $a,b \not\parallel a,c$ and $a,c \parallel p,r$ and $b,c \parallel p,r$, then $p=r$.

(51) If $p,q \not\parallel p,r$ and $p,r \parallel p,s$ and $q,r \parallel q,s$, then $r=s$.

(52) If $a,b \not\parallel a,c$ and $a,b \parallel p,q$ and $a,c \parallel p,r$ and $a,c \parallel p,s$ and $b,c \parallel q,r$ and $b,c \parallel q,s$, then $r=s$.

(53) If $a,b \parallel a,c$ and $a,b \parallel a,d$, then $a,b \parallel c,d$.

(54) If for all a, b holds $a=b$, then for all p, q, r, s holds $p,q \parallel r,s$.

(55) If there exist a, b such that $a \neq b$ and for every c holds $a,b \parallel a,c$, then for all p, q, r, s holds $p,q \parallel r,s$.

(56) If $a,b \not\parallel a,c$ and $p \neq q$, then $p,q \not\parallel p,a$ or $p,q \not\parallel p,b$ or $p,q \not\parallel p,c$.

REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Eugeniusz Kusak. A new approach to dimension-free affine geometry. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 27(11–12):875–882, 1979.

[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[6] Wanda Szmielew. *From Affine to Euclidean Geometry*, volume 27. PWN – D.Reidel Publ. Co., Warszawa – Dordrecht, 1983.

[7] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/domain_1.html.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[6] The proposition (46) has been removed.

[9] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/mcart_1.html`.

[10] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[11] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

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