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Construction of a bilinear symmetric form in orthogonal vector space[1]

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Summary. In this text we present unpublished results by Eugeniusz Kusak and Wojciech Leończuk. They contain an axiomatic description of the class of all spaces $\langle V;\, \perp_{\xi}\rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear symmetric form i.e. $\xi(x,y) = \xi(y,x)$ and $x \perp_{\xi} y$ iff $\xi(x,y) = 0$ for $x, y \in V$. They also contain an effective construction of bilinear symmetric form ξ for given orthogonal space $\langle V;\, \perp\rangle$ such that $\perp = \perp_{\xi}$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a,b,x)$ for $a,b,x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y+\varepsilon z \,\&\, y \perp z+\varepsilon x \Rightarrow z \perp x+\varepsilon y$. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry - see [5].

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The articles [6], [3], [8], [1], [2], [7], [4], [9], and [5] provide the notation and terminology for this paper.

In this paper F is a field.

Let us consider F and let I_1 be an Abelian add-associative right zeroed right complementable non empty simplectic structure over F. We say that I_1 is orthogonality space-like if and only if the condition (Def. 2) is satisfied.

(Def. 2)[1] Let a, b, c, d, x be elements of I_1 and l be an element of F. Then

- (i) if $a \neq 0_{(I_1)}$ and $b \neq 0_{(I_1)}$ and $c \neq 0_{(I_1)}$ and $d \neq 0_{(I_1)}$, then there exists an element p of I_1 such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$,
- (ii) if $a \perp b$, then $l \cdot a \perp b$,
- (iii) if $b \perp a$ and $c \perp a$, then $b + c \perp a$,
- (iv) if $b \not\perp a$, then there exists an element k of F such that $x - k \cdot b \perp a$, and
- (v) if $a \perp b - c$ and $b \perp c - a$, then $c \perp a - b$.

Let us consider F. Observe that there exists an Abelian add-associative right zeroed right complementable non empty simplectic structure over F which is orthogonality space-like, vector space-like, and strict.

Let us consider F. An orthogonality space over F is an orthogonality space-like vector space-like Abelian add-associative right zeroed right complementable non empty simplectic structure over F.

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[1] The definition (Def. 1) has been removed.

We use the following convention: S is an orthogonality space over F, a, b, c, d, p, q, x, y, z are elements of S, and k, l are elements of F.

We now state a number of propositions:

(11)[2] $0_S \perp a$.

(12) If $a \perp b$, then $b \perp a$.

(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.

(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.

(15) If $b \not\perp a$ and $l \neq 0_F$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.

(16) If $a \perp b$, then $-a \perp b$.

(19)[3] If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.

(20) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.

(21) If $a \perp a$ and $b \perp b$, then $a + b \perp a - b$.

(22) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and there exists a such that $a \neq 0_S$, then there exists b such that $b \not\perp b$.

Let us consider F, S, a, b, x. Let us assume that $b \not\perp a$. The functor $\mathrm{J}(a,b,x)$ yielding an element of F is defined as follows:

(Def. 6)[4] For every element l of F such that $x - l \cdot b \perp a$ holds $\mathrm{J}(a,b,x) = l$.

The following propositions are true:

(24)[5] If $b \not\perp a$, then $x - \mathrm{J}(a,b,x) \cdot b \perp a$.

(25) If $b \not\perp a$, then $\mathrm{J}(a,b,l \cdot x) = l \cdot \mathrm{J}(a,b,x)$.

(26) If $b \not\perp a$, then $\mathrm{J}(a,b,x+y) = \mathrm{J}(a,b,x) + \mathrm{J}(a,b,y)$.

(27) If $b \not\perp a$ and $l \neq 0_F$, then $\mathrm{J}(a,l \cdot b,x) = l^{-1} \cdot \mathrm{J}(a,b,x)$.

(28) If $b \not\perp a$ and $l \neq 0_F$, then $\mathrm{J}(l \cdot a,b,x) = \mathrm{J}(a,b,x)$.

(29) If $b \not\perp a$ and $p \perp a$, then $\mathrm{J}(a,b+p,c) = \mathrm{J}(a,b,c)$ and $\mathrm{J}(a,b,c+p) = \mathrm{J}(a,b,c)$.

(30) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $\mathrm{J}(a+p,b,c) = \mathrm{J}(a,b,c)$.

(31) If $b \not\perp a$ and $c - b \perp a$, then $\mathrm{J}(a,b,c) = \mathbf{1}_F$.

(32) If $b \not\perp a$, then $\mathrm{J}(a,b,b) = \mathbf{1}_F$.

(33) If $b \not\perp a$, then $x \perp a$ iff $\mathrm{J}(a,b,x) = 0_F$.

(34) If $b \not\perp a$ and $q \not\perp a$, then $\mathrm{J}(a,b,p) \cdot \mathrm{J}(a,b,q)^{-1} = \mathrm{J}(a,q,p)$.

(35) If $b \not\perp a$ and $c \not\perp a$, then $\mathrm{J}(a,b,c) = \mathrm{J}(a,c,b)^{-1}$.

(36) If $b \not\perp a$ and $b \perp c + a$, then $\mathrm{J}(a,b,c) = -\mathrm{J}(c,b,a)$.

(37) If $a \not\perp b$ and $c \not\perp b$, then $\mathrm{J}(c,b,a) = \mathrm{J}(b,a,c)^{-1} \cdot \mathrm{J}(a,b,c)$.

(38) If $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $\mathrm{J}(a,q,p) \cdot \mathrm{J}(p,a,x) = \mathrm{J}(q,a,x) \cdot \mathrm{J}(x,q,p)$.

[2] The propositions (1)–(10) have been removed.
[3] The propositions (17) and (18) have been removed.
[4] The definitions (Def. 3)–(Def. 5) have been removed.
[5] The proposition (23) has been removed.

(39) If $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$, then $\mathrm{J}(a,b,p) \cdot \mathrm{J}(p,a,x) \cdot \mathrm{J}(x,p,y) = \mathrm{J}(a,b,q) \cdot \mathrm{J}(q,a,x) \cdot \mathrm{J}(x,q,y)$.

(40) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $\mathrm{J}(p,a,x) \cdot \mathrm{J}(x,p,y) = \mathrm{J}(p,a,y) \cdot \mathrm{J}(y,p,x)$.

Let us consider F, S, x, y, a, b. Let us assume that $b \not\perp a$. The functor $x \cdot_{a,b} y$ yields an element of F and is defined as follows:

(Def. 7)(i) For every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = \mathrm{J}(a,b,q) \cdot \mathrm{J}(q,a,x) \cdot \mathrm{J}(x,q,y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$,

(ii) $x \cdot_{a,b} y = 0_F$ if for every p holds $p \perp a$ or $p \perp x$.

We now state several propositions:

(43)[6] If $b \not\perp a$ and $x = 0_S$, then $x \cdot_{a,b} y = 0_F$.

(44) If $b \not\perp a$, then $x \cdot_{a,b} y = 0_F$ iff $y \perp x$.

(45) If $b \not\perp a$, then $x \cdot_{a,b} y = y \cdot_{a,b} x$.

(46) If $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(47) If $b \not\perp a$, then $x \cdot_{a,b} (y+z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

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[6] The propositions (41) and (42) have been removed.