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Kuratowski - Zorn Lemma[1]

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Summary. The goal of this article is to prove Kuratowski - Zorn lemma. We prove it in a number of forms (theorems and schemes). We introduce the following notions: a relation is a quasi (or partial, or linear) order, a relation quasi (or partially, or linearly) orders a set, minimal and maximal element in a relation, inferior and superior element of a relation, a set has lower (or upper) Zorn property w.r.t. a relation. We prove basic theorems concerning those notions and theorems that relate them to the notions introduced in [8]. At the end of the article we prove some theorems that belong rather to [10], [12] or [2].

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The articles [7], [5], [9], [10], [3], [12], [11], [4], [2], [6], [8], and [1] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: R, P denote binary relations, X, X_1, Y, Z, x, y denote sets, O denotes an order in X, A denotes a non empty poset, C denotes a chain of A, S denotes a subset of A, and a, b denote elements of A.

We now state two propositions:

(1) $\mathrm{dom}\, O = X$ and $\mathrm{rng}\, O = X$.

(2) $\mathrm{field}\, O = X$.

Let us consider R. We say that R is quasi-order if and only if:

(Def. 1) R is reflexive and transitive.

We introduce R is a quasi order as a synonym of R is quasi-order. We say that R is partial-order if and only if:

(Def. 2) R is reflexive, transitive, and antisymmetric.

We introduce R is a partial order as a synonym of R is partial-order. We say that R is linear-order if and only if:

(Def. 3) R is reflexive, transitive, antisymmetric, and connected.

We introduce R is a linear order as a synonym of R is linear-order.

We now state a number of propositions:

(6)[1] If R is a quasi order, then $R^{\smile}$ is a quasi order.

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[1]Supported by RPBP.III-24.C1.

[1] The propositions (3)–(5) have been removed.

(7) If R is a partial order, then $R^\smile$ is a partial order.

(8) If R is a linear order, then $R^\smile$ is a linear order.

(9) If R is well-ordering, then R is a quasi order, a partial order, and a linear order.

(10) If R is a linear order, then R is a quasi order and a partial order.

(11) If R is a partial order, then R is a quasi order.

(12) O is a partial order.

(13) O is a quasi order.

(14) If O is connected, then O is a linear order.

(15) If R is a quasi order, then $R\,|^2\,X$ is a quasi order.

(16) If R is a partial order, then $R\,|^2\,X$ is a partial order.

(17) If R is a linear order, then $R\,|^2\,X$ is a linear order.

(18) field((the internal relation of A) $|^2\,S) = S$.

(19) If (the internal relation of A) $|^2\,S$ is a linear order, then S is a chain of A.

(20) (The internal relation of A) $|^2\,C$ is a linear order.

(21) $\emptyset$ is a quasi order and $\emptyset$ is a partial order and $\emptyset$ is a linear order and $\emptyset$ is well-ordering.

(22) id_X is a quasi order and id_X is a partial order.

Let us consider R, X. We say that R quasi orders X if and only if:

(Def. 4) R is reflexive in X and transitive in X.

We say that R partially orders X if and only if:

(Def. 5) R is reflexive in X, transitive in X, and antisymmetric in X.

We say that R linearly orders X if and only if:

(Def. 6) R is reflexive in X, transitive in X, antisymmetric in X, and connected in X.

One can prove the following propositions:

(26)[2] If R well orders X, then R quasi orders X and R partially orders X and R linearly orders X.

(27) If R linearly orders X, then R quasi orders X and R partially orders X.

(28) If R partially orders X, then R quasi orders X.

(29) If R is a quasi order, then R quasi orders field R.

(30) If R quasi orders Y and $X \subseteq Y$, then R quasi orders X.

(31) If R quasi orders X, then $R\,|^2\,X$ is a quasi order.

(32) If R is a partial order, then R partially orders field R.

(33) If R partially orders Y and $X \subseteq Y$, then R partially orders X.

(34) If R partially orders X, then $R\,|^2\,X$ is a partial order.

(35) If R is a linear order, then R linearly orders field R.

[2] The propositions (23)–(25) have been removed.

(36) If R linearly orders Y and $X \subseteq Y$, then R linearly orders X.

(37) If R linearly orders X, then $R\,|^2\,X$ is a linear order.

(38) If R quasi orders X, then $R^\smile$ quasi orders X.

(39) If R partially orders X, then $R^\smile$ partially orders X.

(40) If R linearly orders X, then $R^\smile$ linearly orders X.

(41) O quasi orders X.

(42) O partially orders X.

(43) If R partially orders X, then $R\,|^2\,X$ is an order in X.

(44) If R linearly orders X, then $R\,|^2\,X$ is an order in X.

(45) If R well orders X, then $R\,|^2\,X$ is an order in X.

(46) If the internal relation of A linearly orders S, then S is a chain of A.

(47) The internal relation of A linearly orders C.

(48) id_X quasi orders X and id_X partially orders X.

Let us consider R, X. We say that X has the upper Zorn property w.r.t. R if and only if:

(Def. 7) For every Y such that $Y \subseteq X$ and $R\,|^2\,Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle y, x\rangle \in R$.

We say that X has the lower Zorn property w.r.t. R if and only if:

(Def. 8) For every Y such that $Y \subseteq X$ and $R\,|^2\,Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle x, y\rangle \in R$.

We now state four propositions:

(51)[3] If X has the upper Zorn property w.r.t. R, then $X \neq \emptyset$.

(52) If X has the lower Zorn property w.r.t. R, then $X \neq \emptyset$.

(53) X has the upper Zorn property w.r.t. R iff X has the lower Zorn property w.r.t. $R^\smile$.

(54) X has the upper Zorn property w.r.t. $R^\smile$ iff X has the lower Zorn property w.r.t. R.

Let us consider R, x. We say that x is maximal in R if and only if:

(Def. 9) $x \in \mathrm{field}\,R$ and it is not true that there exists y such that $y \in \mathrm{field}\,R$ and $y \neq x$ and $\langle x, y\rangle \in R$.

We say that x is minimal in R if and only if:

(Def. 10) $x \in \mathrm{field}\,R$ and it is not true that there exists y such that $y \in \mathrm{field}\,R$ and $y \neq x$ and $\langle y, x\rangle \in R$.

We say that x is superior of R if and only if:

(Def. 11) $x \in \mathrm{field}\,R$ and for every y such that $y \in \mathrm{field}\,R$ and $y \neq x$ holds $\langle y, x\rangle \in R$.

We say that x is inferior of R if and only if:

(Def. 12) $x \in \mathrm{field}\,R$ and for every y such that $y \in \mathrm{field}\,R$ and $y \neq x$ holds $\langle x, y\rangle \in R$.

One can prove the following propositions:

[3] The propositions (49) and (50) have been removed.

(59)[4] If x is inferior of R and R is antisymmetric, then x is minimal in R.

(60) If x is superior of R and R is antisymmetric, then x is maximal in R.

(61) If x is minimal in R and R is connected, then x is inferior of R.

(62) If x is maximal in R and R is connected, then x is superior of R.

(63) If $x \in X$ and x is superior of R and $X \subseteq \mathrm{field}\,R$ and R is reflexive, then X has the upper Zorn property w.r.t. R.

(64) If $x \in X$ and x is inferior of R and $X \subseteq \mathrm{field}\,R$ and R is reflexive, then X has the lower Zorn property w.r.t. R.

(65) x is minimal in R iff x is maximal in $R^{\smile}$.

(66) x is minimal in $R^{\smile}$ iff x is maximal in R.

(67) x is inferior of R iff x is superior of $R^{\smile}$.

(68) x is inferior of $R^{\smile}$ iff x is superior of R.

(69) a is minimal in the internal relation of A iff for every b holds $b \not< a$.

(70) a is maximal in the internal relation of A iff for every b holds $a \not< b$.

(71) a is superior of the internal relation of A iff for every b such that $a \neq b$ holds $b < a$.

(72) a is inferior of the internal relation of A iff for every b such that $a \neq b$ holds $a < b$.

(73) If for every C there exists a such that for every b such that $b \in C$ holds $b \leq a$, then there exists a such that for every b holds $a \not< b$.

Let A be a non empty set and let O be an order in A. One can verify that $\langle A, O \rangle$ is non empty.
We now state several propositions:

(74) If for every C there exists a such that for every b such that $b \in C$ holds $a \leq b$, then there exists a such that for every b holds $b \not< a$.

(75) For all R, X such that R partially orders X and $\mathrm{field}\,R = X$ and X has the upper Zorn property w.r.t. R holds there exists x which is maximal in R.

(76) For all R, X such that R partially orders X and $\mathrm{field}\,R = X$ and X has the lower Zorn property w.r.t. R holds there exists x which is minimal in R.

(77) Let given X. Suppose that

(i) $X \neq \emptyset$, and

(ii) for every Z such that $Z \subseteq X$ and Z is $\subseteq$-linear there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

(78) Let given X. Suppose that

(i) $X \neq \emptyset$, and

(ii) for every Z such that $Z \subseteq X$ and Z is $\subseteq$-linear there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $Y \subseteq X_1$.

Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Z \not\subseteq Y$.

(79) Let given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and Z is $\subseteq$-linear holds $\bigcup Z \in X$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

[4] The propositions (55)–(58) have been removed.

(80) Let given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and Z is $\subseteq$-linear holds $\bigcap Z \in X$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Z \not\subseteq Y$.

In this article we present several logical schemes. The scheme *Zorn Max* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists an element x of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[x,y]$

provided the following conditions are met:

- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x,x]$,
- For all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,x]$ holds $x = y$,
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,z]$ holds $\mathcal{P}[x,z]$, and
- Let given X. Suppose $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x,y]$ or $\mathcal{P}[y,x]$. Then there exists an element y of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[x,y]$.

The scheme *Zorn Min* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists an element x of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[y,x]$

provided the parameters meet the following conditions:

- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x,x]$,
- For all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,x]$ holds $x = y$,
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,z]$ holds $\mathcal{P}[x,z]$, and
- Let given X. Suppose $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x,y]$ or $\mathcal{P}[y,x]$. Then there exists an element y of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[y,x]$.

We now state a number of propositions:

(81) If R partially orders X and $\mathrm{field}\,R = X$, then there exists P such that $R \subseteq P$ and P linearly orders X and $\mathrm{field}\,P = X$.

(82) $R \subseteq [:\mathrm{field}\,R, \mathrm{field}\,R:]$.

(83) If R is reflexive and $X \subseteq \mathrm{field}\,R$, then $\mathrm{field}(R\,|^2\,X) = X$.

(84) If R is reflexive in X, then $R\,|^2\,X$ is reflexive.

(85) If R is transitive in X, then $R\,|^2\,X$ is transitive.

(86) If R is antisymmetric in X, then $R\,|^2\,X$ is antisymmetric.

(87) If R is connected in X, then $R\,|^2\,X$ is connected.

(88) If R is connected in X and $Y \subseteq X$, then R is connected in Y.

(89) If R well orders X and $Y \subseteq X$, then R well orders Y.

(90) If R is connected, then $R^\smile$ is connected.

(91) If R is reflexive in X, then $R^\smile$ is reflexive in X.

(92) If R is transitive in X, then $R^\smile$ is transitive in X.

(93) If R is antisymmetric in X, then $R^\smile$ is antisymmetric in X.

(94) If R is connected in X, then $R^\smile$ is connected in X.

(95) $(R\,|^2\,X)^\smile = R^\smile\,|^2\,X$.

(96) $R\,|^2\,\emptyset = \emptyset$.

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