JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Partially Ordered Sets

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Summary. In the beginning of this article we define the choice function of a non-empty set family that does not contain $\emptyset$ as introduced in [6, pages 88–89]. We define order of a set as a relation being reflexive, antisymmetric and transitive in the set, partially ordered set as structure non-empty set and order of the set, chains, lower and upper cone of a subset, initial segments of element and subset of partially ordered set. Some theorems that belong rather to [5] or [12] are proved.

MML Identifier: ORDERS_1.

WWW: http://mizar.org/JFM/Vol1/orders_1.html

The articles [8], [5], [9], [10], [12], [2], [11], [4], [3], [1], and [7] provide the notation and terminology for this paper.

We adopt the following convention: X, Y denote sets, x, y, z denote sets, and M denotes a non empty set.

Let us consider M. Let us assume that $\emptyset \notin M$. A function from M into $\bigcup M$ is said to be a choice function of M if:

(Def. 1) For every X such that $X \in M$ holds $\mathrm{it}(X) \in X$.

In the sequel D, D_1 denote non empty sets.

Let D be a set. The functor 2^D_+ yields a set and is defined by:

(Def. 2) $2^D_+ = 2^D \setminus \{\emptyset\}$.

Let us consider D. One can verify that 2^D_+ is non empty.

Next we state four propositions:

(4)[1] $\emptyset \notin 2^D_+$.

(5) $D_1 \subseteq D$ iff $D_1 \in 2^D_+$.

(6) D_1 is a subset of D iff $D_1 \in 2^D_+$.

(7) $D \in 2^D_+$.

In the sequel P is a binary relation.

Let us consider X. An order in X is a total reflexive antisymmetric transitive binary relation on X.

In the sequel O is an order in X.

We now state three propositions:

(12)[2] If $x \in X$, then $\langle x, x \rangle \in O$.

[1] The propositions (1)–(3) have been removed.
[2] The propositions (8)–(11) have been removed.

(13) If $x \in X$ and $y \in X$ and $\langle x, y \rangle \in O$ and $\langle y, x \rangle \in O$, then $x = y$.

(14) If $x \in X$ and $y \in X$ and $z \in X$ and $\langle x, y \rangle \in O$ and $\langle y, z \rangle \in O$, then $\langle x, z \rangle \in O$.

We consider relational structures as extensions of 1-sorted structure as systems
⟨ a carrier, an internal relation ⟩,
where the carrier is a set and the internal relation is a binary relation on the carrier.

Let X be a non empty set and let R be a binary relation on X. Observe that $\langle X, R \rangle$ is non empty.

Let A be a relational structure. We say that A is reflexive if and only if:

(Def. 4)[3] The internal relation of A is reflexive in the carrier of A.

We say that A is transitive if and only if:

(Def. 5) The internal relation of A is transitive in the carrier of A.

We say that A is antisymmetric if and only if:

(Def. 6) The internal relation of A is antisymmetric in the carrier of A.

Let us note that there exists a relational structure which is non empty, reflexive, transitive, antisymmetric, and strict.

A poset is a reflexive transitive antisymmetric relational structure.

Let A be a poset. Note that the internal relation of A is total, reflexive, antisymmetric, and transitive.

Let X be a set and let O be an order in X. One can verify that $\langle X, O \rangle$ is reflexive, transitive, and antisymmetric.

We use the following convention: A denotes a non empty poset, a, a_1, a_2, b, c denote elements of A, and S, T denote subsets of A.

Let A be a relational structure and let a_1, a_2 be elements of A. The predicate $a_1 \leq a_2$ is defined by:

(Def. 9)[4] $\langle a_1, a_2 \rangle \in$ the internal relation of A.

We introduce $a_2 \geq a_1$ as a synonym of $a_1 \leq a_2$.

Let A be a relational structure and let a_1, a_2 be elements of A. The predicate $a_1 < a_2$ is defined as follows:

(Def. 10) $a_1 \leq a_2$ and $a_1 \neq a_2$.

Let us note that the predicate $a_1 < a_2$ is irreflexive. We introduce $a_2 > a_1$ as a synonym of $a_1 < a_2$.

We now state the proposition

(24)[5] For every reflexive non empty relational structure A and for every element a of A holds $a \leq a$.

Let A be a reflexive non empty relational structure and let a_1, a_2 be elements of A. Let us note that the predicate $a_1 \leq a_2$ is reflexive.

One can prove the following propositions:

(25) For every antisymmetric relational structure A and for all elements a_1, a_2 of A such that $a_1 \leq a_2$ and $a_2 \leq a_1$ holds $a_1 = a_2$.

(26) For every transitive relational structure A and for all elements a_1, a_2, a_3 of A such that $a_1 \leq a_2$ and $a_2 \leq a_3$ holds $a_1 \leq a_3$.

(28)[6] For every antisymmetric relational structure A and for all elements a_1, a_2 of A holds $a_1 \not< a_2$ or $a_2 \not< a_1$.

[3] The definition (Def. 3) has been removed.
[4] The definitions (Def. 7) and (Def. 8) have been removed.
[5] The propositions (15)–(23) have been removed.
[6] The proposition (27) has been removed.

(29) Let A be a transitive antisymmetric relational structure and a_1, a_2, a_3 be elements of A. If $a_1 < a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

(30) For every antisymmetric relational structure A and for all elements a_1, a_2 of A such that $a_1 \leq a_2$ holds $a_2 \not< a_1$.

(32)[7] Let A be a transitive antisymmetric relational structure and a_1, a_2, a_3 be elements of A. If $a_1 < a_2$ and $a_2 \leq a_3$ or $a_1 \leq a_2$ and $a_2 < a_3$, then $a_1 < a_3$.

Let A be a relational structure and let I_1 be a subset of A. We say that I_1 is strongly connected if and only if:

(Def. 11) The internal relation of A is strongly connected in I_1.

Let A be a relational structure. Note that $\emptyset_A$ is strongly connected.
Let A be a relational structure. Note that there exists a subset of A which is strongly connected.
Let A be a relational structure. A chain of A is a strongly connected subset of A.
The following propositions are true:

(35)[8] For every non empty reflexive relational structure A and for every element a of A holds $\{a\}$ is a chain of A.

(36) Let A be a non empty reflexive relational structure and a_1, a_2 be elements of A. Then $\{a_1, a_2\}$ is a chain of A if and only if $a_1 \leq a_2$ or $a_2 \leq a_1$.

(37) Let A be a relational structure, C be a chain of A, and S be a subset of A. If $S \subseteq C$, then S is a chain of A.

(38) Let A be a reflexive relational structure and a_1, a_2 be elements of A. Then there exists a chain C of A such that $a_1 \in C$ and $a_2 \in C$ if and only if $a_1 \leq a_2$ or $a_2 \leq a_1$.

(39) Let A be a reflexive antisymmetric relational structure and a_1, a_2 be elements of A. Then there exists a chain C of A such that $a_1 \in C$ and $a_2 \in C$ if and only if $a_1 < a_2$ iff $a_2 \not\leq a_1$.

(40) Let A be a relational structure and T be a subset of A. Suppose the internal relation of A well orders T. Then T is a chain of A.

Let us consider A and let us consider S. The functor $\mathrm{UpperCone}\,S$ yields a subset of A and is defined by:

(Def. 12) $\mathrm{UpperCone}\,S = \{a_1 : \bigwedge_{a_2} (a_2 \in S \Rightarrow a_2 < a_1)\}$.

Let us consider A and let us consider S. The functor $\mathrm{LowerCone}\,S$ yielding a subset of A is defined as follows:

(Def. 13) $\mathrm{LowerCone}\,S = \{a_1 : \bigwedge_{a_2} (a_2 \in S \Rightarrow a_1 < a_2)\}$.

One can prove the following propositions:

(43)[9] $\mathrm{UpperCone}(\emptyset_A) =$ the carrier of A.

(44) $\mathrm{UpperCone}(\Omega_A) = \emptyset$.

(45) $\mathrm{LowerCone}(\emptyset_A) =$ the carrier of A.

(46) $\mathrm{LowerCone}(\Omega_A) = \emptyset$.

(47) If $a \in S$, then $a \notin \mathrm{UpperCone}\,S$.

(48) $a \notin \mathrm{UpperCone}\{a\}$.

[7] The proposition (31) has been removed.
[8] The propositions (33) and (34) have been removed.
[9] The propositions (41) and (42) have been removed.

(49) If $a \in S$, then $a \notin \operatorname{LowerCone} S$.

(50) $a \notin \operatorname{LowerCone}\{a\}$.

(51) $c < a$ iff $a \in \operatorname{UpperCone}\{c\}$.

(52) $a < c$ iff $a \in \operatorname{LowerCone}\{c\}$.

Let us consider A, let us consider S, and let us consider a. The functor $\operatorname{InitSegm}(S,a)$ yields a subset of A and is defined by:

(Def. 14) $\operatorname{InitSegm}(S,a) = \operatorname{LowerCone}\{a\} \cap S$.

Let us consider A and let us consider S. A subset of A is called an initial segment of S if:

(Def. 15)(i) There exists a such that $a \in S$ and it $= \operatorname{InitSegm}(S,a)$ if $S \neq \emptyset$,

(ii) it $= \emptyset$, otherwise.

In the sequel I denotes an initial segment of S.
One can prove the following propositions:

(56)[10] $x \in \operatorname{InitSegm}(S,a)$ iff $x \in \operatorname{LowerCone}\{a\}$ and $x \in S$.

(57) $a \in \operatorname{InitSegm}(S,b)$ iff $a < b$ and $a \in S$.

(60)[11] $\operatorname{InitSegm}(\emptyset_A, a) = \emptyset$.

(61) $\operatorname{InitSegm}(S,a) \subseteq S$.

(62) $a \notin \operatorname{InitSegm}(S,a)$.

(64)[12] If $a_1 < a_2$, then $\operatorname{InitSegm}(S,a_1) \subseteq \operatorname{InitSegm}(S,a_2)$.

(65) If $S \subseteq T$, then $\operatorname{InitSegm}(S,a) \subseteq \operatorname{InitSegm}(T,a)$.

(67)[13] $I \subseteq S$.

(68) $S \neq \emptyset$ iff S is not an initial segment of S.

(69) If $S \neq \emptyset$ or $T \neq \emptyset$ and if S is an initial segment of T, then T is not an initial segment of S.

(70) If $a_1 < a_2$ and $a_1 \in S$ and $a_2 \in T$ and T is an initial segment of S, then $a_1 \in T$.

(71) If $a \in S$ and S is an initial segment of T, then $\operatorname{InitSegm}(S,a) = \operatorname{InitSegm}(T,a)$.

(72) Suppose $S \subseteq T$ and the internal relation of A well orders T and for all a_1, a_2 such that $a_2 \in S$ and $a_1 < a_2$ holds $a_1 \in S$. Then $S = T$ or S is an initial segment of T.

(73) Suppose $S \subseteq T$ and the internal relation of A well orders T and for all a_1, a_2 such that $a_2 \in S$ and $a_1 \in T$ and $a_1 < a_2$ holds $a_1 \in S$. Then $S = T$ or S is an initial segment of T.

In the sequel f denotes a choice function of $2_{+}^{\text{the carrier of } A}$.
Let us consider A and let us consider f. A chain of A is called a chain of f if:

(Def. 16) It $\neq \emptyset$ and the internal relation of A well orders it and for every a such that $a \in$ it holds $f(\operatorname{UpperCone} \operatorname{InitSegm}(\text{it}, a)) = a$.

In the sequel f_1, f_2, f_3 are chains of f.
One can prove the following propositions:

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[10] The propositions (53)–(55) have been removed.
[11] The propositions (58) and (59) have been removed.
[12] The proposition (63) has been removed.
[13] The proposition (66) has been removed.

(78)[14] $\{f(\text{the carrier of } A)\}$ is a chain of f.

(79) $f(\text{the carrier of } A) \in f_1$.

(80) If $a \in f_1$ and $b = f(\text{the carrier of } A)$, then $b \leq a$.

(81) If $a = f(\text{the carrier of } A)$, then $\mathrm{InitSegm}(f_1, a) = \emptyset$.

(82) f_2 meets f_3.

(83) If $f_2 \neq f_3$, then f_2 is an initial segment of f_3 iff f_3 is not an initial segment of f_2.

(84) $f_2 \subset f_3$ iff f_2 is an initial segment of f_3.

Let us consider A and let us consider f. The functor $\mathrm{Chains}\, f$ yields a set and is defined by:

(Def. 17) $x \in \mathrm{Chains}\, f$ iff x is a chain of f.

Let us consider A and let us consider f. Note that $\mathrm{Chains}\, f$ is non empty.
Next we state a number of propositions:

(87)[15] $\bigcup \mathrm{Chains}\, f \neq \emptyset$.

(88) If $f_1 \neq \bigcup \mathrm{Chains}\, f$ and $S = \bigcup \mathrm{Chains}\, f$, then f_1 is an initial segment of S.

(89) $\bigcup \mathrm{Chains}\, f$ is a chain of f.

(91)[16] There exists X such that $X \neq \emptyset$ and $X \in Y$ iff $\bigcup Y \neq \emptyset$.

(92) P is strongly connected in X iff P is reflexive in X and connected in X.

(93) If P is reflexive in X and $Y \subseteq X$, then P is reflexive in Y.

(94) If P is antisymmetric in X and $Y \subseteq X$, then P is antisymmetric in Y.

(95) If P is transitive in X and $Y \subseteq X$, then P is transitive in Y.

(96) If P is strongly connected in X and $Y \subseteq X$, then P is strongly connected in Y.

(97) For every total reflexive binary relation R on X holds $\mathrm{field}\, R = X$.

(98) For every set A and for every binary relation R on A such that R is reflexive in A holds $\mathrm{dom}\, R = A$ and $\mathrm{field}\, R = A$.

REFERENCES

[1] Grzegorz Bancerek. The well ordering relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/wellord1.html`.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[4] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[6] Kazimierz Kuratowski. *Wstęp do teorii mnogości i topologii*. PWN, Warszawa, 1977.

[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[14] The propositions (74)–(77) have been removed.
[15] The propositions (85) and (86) have been removed.
[16] The proposition (90) has been removed.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[9] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[10] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[11] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[12] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

Received August 30, 1989

Published January 2, 2004