JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

Representation Theorem for Heyting Lattices

Jolanta Kamieńska
Warsaw University
Białystok

MML Identifier: OPENLATT.

WWW: http://mizar.org/JFM/Vol5/openlatt.html

The articles [11], [7], [13], [1], [14], [5], [6], [4], [9], [10], [15], [16], [12], [2], [3], and [8] provide the notation and terminology for this paper.

Let us observe that every lower bound lattice which is Heyting is also implicative and every lattice which is implicative is also upper-bounded.

In the sequel T is a topological space and A, B are subsets of T.

We now state two propositions:

(1) $A \cap \mathrm{Int}(A^{\mathrm{c}} \cup B) \subseteq B$.

(2) For every subset C of T such that C is open and $A \cap C \subseteq B$ holds $C \subseteq \mathrm{Int}(A^{\mathrm{c}} \cup B)$.

Let T be a topological structure. The functor $\mathrm{Topology}(T)$ yields a family of subsets of T and is defined as follows:

(Def. 1) $\mathrm{Topology}(T) =$ the topology of T.

Let us consider T. Note that $\mathrm{Topology}(T)$ is non empty.

One can prove the following proposition

(3) For every subset A of T holds A is open iff $A \in \mathrm{Topology}(T)$.

Let T be a non empty topological space and let P, Q be elements of $\mathrm{Topology}(T)$. Then $P \cup Q$ is an element of $\mathrm{Topology}(T)$. Then $P \cap Q$ is an element of $\mathrm{Topology}(T)$.

In the sequel T denotes a non empty topological space and P, Q denote elements of $\mathrm{Topology}(T)$.

Let us consider T. The functor $\mathrm{TopUnion}(T)$ yielding a binary operation on $\mathrm{Topology}(T)$ is defined as follows:

(Def. 2) $(\mathrm{TopUnion}(T))(P, Q) = P \cup Q$.

The functor $\mathrm{TopMeet}(T)$ yielding a binary operation on $\mathrm{Topology}(T)$ is defined by:

(Def. 3) $(\mathrm{TopMeet}(T))(P, Q) = P \cap Q$.

One can prove the following proposition

(4) For every non empty topological space T holds $\langle \mathrm{Topology}(T), \mathrm{TopUnion}(T), \mathrm{TopMeet}(T) \rangle$ is a lattice.

Let us consider T. The functor $\mathrm{OpenSetLatt}(T)$ yielding a lattice is defined by:

(Def. 4) $\text{OpenSetLatt}(T) = \langle \text{Topology}(T), \text{TopUnion}(T), \text{TopMeet}(T) \rangle$.

The following proposition is true

(5) The carrier of $\text{OpenSetLatt}(T) = \text{Topology}(T)$.

In the sequel p, q denote elements of $\text{OpenSetLatt}(T)$.
We now state several propositions:

(6) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(7) $p \sqsubseteq q$ iff $p \subseteq q$.

(8) For all elements p', q' of $\text{Topology}(T)$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \subseteq q'$.

(9) $\text{OpenSetLatt}(T)$ is implicative.

(10) $\text{OpenSetLatt}(T)$ is lower-bounded and $\bot_{\text{OpenSetLatt}(T)} = \emptyset$.

Let us consider T. Observe that $\text{OpenSetLatt}(T)$ is Heyting.
The following proposition is true

(11) $\top_{\text{OpenSetLatt}(T)} =$ the carrier of T.

For simplicity, we adopt the following convention: L is a distributive lattice, F is a filter of L, a, b are elements of L, and x, X_1, Y, Z are sets.
Let us consider L. The functor $\text{PrimeFilters}(L)$ yields a set and is defined as follows:

(Def. 5) $\text{PrimeFilters}(L) = \{F : F \neq \text{the carrier of } L \wedge F \text{is prime}\}$.

We now state the proposition

(12) $F \in \text{PrimeFilters}(L)$ iff $F \neq$ the carrier of L and F is prime.

Let us consider L. The functor $\text{StoneH}(L)$ yields a function and is defined by:

(Def. 6) $\text{dom}\,\text{StoneH}(L) =$ the carrier of L and $(\text{StoneH}(L))(a) = \{F : F \in \text{PrimeFilters}(L) \wedge a \in F\}$.

Next we state two propositions:

(13) $F \in (\text{StoneH}(L))(a)$ iff $F \in \text{PrimeFilters}(L)$ and $a \in F$.

(14) $x \in (\text{StoneH}(L))(a)$ iff there exists F such that $F = x$ and $F \neq$ the carrier of L and F is prime and $a \in F$.

Let us consider L. The functor $\text{StoneS}(L)$ yielding a set is defined as follows:

(Def. 7) $\text{StoneS}(L) = \text{rng}\,\text{StoneH}(L)$.

Let us consider L. Note that $\text{StoneS}(L)$ is non empty.
One can prove the following three propositions:

(15) $x \in \text{StoneS}(L)$ iff there exists a such that $x = (\text{StoneH}(L))(a)$.

(16) $(\text{StoneH}(L))(a \sqcup b) = (\text{StoneH}(L))(a) \cup (\text{StoneH}(L))(b)$.

(17) $(\text{StoneH}(L))(a \sqcap b) = (\text{StoneH}(L))(a) \cap (\text{StoneH}(L))(b)$.

Let us consider L, a. The functor $\text{Filters}(a)$ yields a family of subsets of L and is defined as follows:

(Def. 8) $\text{Filters}(a) = \{F : a \in F\}$.

Let us consider L and let us consider a. Observe that $\mathrm{Filters}(a)$ is non empty.
The following propositions are true:

(18) $x \in \mathrm{Filters}(a)$ iff x is a filter of L and $a \in x$.

(19) If $x \in \mathrm{Filters}(b) \setminus \mathrm{Filters}(a)$, then x is a filter of L and $b \in x$ and $a \notin x$.

(20) Let given Z. Suppose $Z \neq \emptyset$ and $Z \subseteq \mathrm{Filters}(b) \setminus \mathrm{Filters}(a)$ and Z is $\subseteq$-linear. Then there exists Y such that $Y \in \mathrm{Filters}(b) \setminus \mathrm{Filters}(a)$ and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

(21) If $b \not\sqsubseteq a$, then $[b) \in \mathrm{Filters}(b) \setminus \mathrm{Filters}(a)$.

(22) If $b \not\sqsubseteq a$, then there exists F such that $F \in \mathrm{PrimeFilters}(L)$ and $a \notin F$ and $b \in F$.

(23) If $a \neq b$, then there exists F such that $F \in \mathrm{PrimeFilters}(L)$.

(24) If $a \neq b$, then $(\mathrm{StoneH}(L))(a) \neq (\mathrm{StoneH}(L))(b)$.

(25) $\mathrm{StoneH}(L)$ is one-to-one.

Let us consider L and let A, B be elements of $\mathrm{StoneS}(L)$. Then $A \cup B$ is an element of $\mathrm{StoneS}(L)$. Then $A \cap B$ is an element of $\mathrm{StoneS}(L)$.

Let us consider L. The functor $\mathrm{SetUnion}(L)$ yielding a binary operation on $\mathrm{StoneS}(L)$ is defined by:

(Def. 9) For all elements A, B of $\mathrm{StoneS}(L)$ holds $(\mathrm{SetUnion}(L))(A, B) = A \cup B$.

The functor $\mathrm{SetMeet}(L)$ yielding a binary operation on $\mathrm{StoneS}(L)$ is defined as follows:

(Def. 10) For all elements A, B of $\mathrm{StoneS}(L)$ holds $(\mathrm{SetMeet}(L))(A, B) = A \cap B$.

Next we state the proposition

(26) $\langle \mathrm{StoneS}(L), \mathrm{SetUnion}(L), \mathrm{SetMeet}(L) \rangle$ is a lattice.

Let us consider L. The functor $\mathrm{StoneLatt}(L)$ yielding a lattice is defined by:

(Def. 11) $\mathrm{StoneLatt}(L) = \langle \mathrm{StoneS}(L), \mathrm{SetUnion}(L), \mathrm{SetMeet}(L) \rangle$.

In the sequel p, q are elements of $\mathrm{StoneLatt}(L)$.
One can prove the following propositions:

(27) For every L holds the carrier of $\mathrm{StoneLatt}(L) = \mathrm{StoneS}(L)$.

(28) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(29) $p \sqsubseteq q$ iff $p \subseteq q$.

Let us consider L. Then $\mathrm{StoneH}(L)$ is a homomorphism from L to $\mathrm{StoneLatt}(L)$.
We now state three propositions:

(30) $\mathrm{StoneH}(L)$ is isomorphism.

(31) $\mathrm{StoneLatt}(L)$ is distributive.

(32) L and $\mathrm{StoneLatt}(L)$ are isomorphic.

Let us note that there exists a Heyting lattice which is non trivial.
In the sequel H is a non trivial Heyting lattice and p', q' are elements of H.
One can prove the following three propositions:

(33) $(\mathrm{StoneH}(H))(\top_H) = \mathrm{PrimeFilters}(H)$.

(34) $(\mathrm{StoneH}(H))(\bot_H) = \emptyset$.

(35) $\mathrm{StoneS}(H) \subseteq 2^{\mathrm{PrimeFilters}(H)}$.

Let us consider H. Observe that $\mathrm{PrimeFilters}(H)$ is non empty.

Let us consider H. The functor $\mathrm{HTopSpace}(H)$ yielding a strict topological structure is defined by:

(Def. 12) The carrier of $\mathrm{HTopSpace}(H) = \mathrm{PrimeFilters}(H)$ and the topology of $\mathrm{HTopSpace}(H) = \{\bigcup A : A$ ranges over subsets of $\mathrm{StoneS}(H)\}$.

Let us consider H. Note that $\mathrm{HTopSpace}(H)$ is non empty and topological space-like.

We now state two propositions:

(36) The carrier of $\mathrm{OpenSetLatt}(\mathrm{HTopSpace}(H)) = \{\bigcup A : A$ ranges over subsets of $\mathrm{StoneS}(H)\}$.

(37) $\mathrm{StoneS}(H) \subseteq$ the carrier of $\mathrm{OpenSetLatt}(\mathrm{HTopSpace}(H))$.

Let us consider H. Then $\mathrm{StoneH}(H)$ is a homomorphism from H to $\mathrm{OpenSetLatt}(\mathrm{HTopSpace}(H))$.

We now state several propositions:

(38) $\mathrm{StoneH}(H)$ is monomorphism.

(39) $(\mathrm{StoneH}(H))(p' \Rightarrow q') = (\mathrm{StoneH}(H))(p') \Rightarrow (\mathrm{StoneH}(H))(q')$.

(40) $\mathrm{StoneH}(H)$ preserves implication.

(41) $\mathrm{StoneH}(H)$ preserves top.

(42) $\mathrm{StoneH}(H)$ preserves bottom.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Grzegorz Bancerek. Filters — part I. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/filter_0.html`.

[3] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/realset1.html`.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Jolanta Kamieńska and Jarosław Stanisław Walijewski. Homomorphisms of lattices, finite join and finite meet. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/lattice4.html`.

[9] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[12] Andrzej Trybulec. Finite join and finite meet, and dual lattices. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/lattice2.html`.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[15] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_1.html`.

[16] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

Received July 14, 1993

Published January 2, 2004