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Ordered Rings - Part I

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Summary. This series of papers is devoted to the notion of the ordered ring, and one of its most important cases: the notion of ordered field. It follows the results of [5]. The idea of the notion of order in the ring is based on that of positive cone i.e. the set of positive elements. Positive cone has to contain at least squares of all elements, and be closed under sum and product. Therefore the key notions of this theory are that of square, sum of squares, product of squares, etc. and finally elements generated from squares by means of sums and products. Part I contains definitions of all those key notions and inclusions between them.

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The articles [6], [8], [1], [3], [2], [7], and [4] provide the notation and terminology for this paper.

For simplicity, we use the following convention: i, j, n denote natural numbers, R denotes a non empty double loop structure, x, y denote scalars of R, and f denotes a finite sequence of elements of the carrier of R.

Let D be a non empty set, let f be a finite sequence of elements of D, and let k be a natural number. Let us assume that $0 \neq k$ and $k \leq \operatorname{len} f$. The functor $f^\circ k$ yielding an element of D is defined as follows:

(Def. 1) $f^\circ k = f(k)$.

Let R be a non empty double loop structure and let x be a scalar of R. The functor $x^{\mathbf{2}}$ yields a scalar of R and is defined by:

(Def. 2) $x^{\mathbf{2}} = x \cdot x$.

Let R be a non empty double loop structure and let x be a scalar of R. We say that x is a square if and only if:

(Def. 3) There exists a scalar y of R such that $x = y^{\mathbf{2}}$.

We introduce x is a square as a synonym of x is a square.

Let us consider R, f. We say that f is sequence of sums of squares if and only if:

(Def. 4) $\operatorname{len} f \neq 0$ and $f^\circ 1$ is a square and for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a square and $f^\circ(n+1) = f^\circ n + y$.

We introduce f is a sequence of sums of squares as a synonym of f is sequence of sums of squares.

Let R be a non empty double loop structure and let x be a scalar of R. We say that x is a sum of squares if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a finite sequence f of elements of the carrier of R such that f is a sequence of sums of squares and $x = f^\circ \operatorname{len} f$.

We introduce x is a sum of squares as a synonym of x is a sum of squares.

Let us consider R, f. We say that f is sequence of products of squares if and only if:

(Def. 6) $\operatorname{len} f \neq 0$ and $f^\circ 1$ is a square and for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a square and $f^\circ(n+1) = f^\circ n \cdot y$.

We introduce f is a sequence of products of squares as a synonym of f is sequence of products of squares.

Let us consider R, x. We say that x is a product of squares if and only if:

(Def. 7) There exists f such that f is a sequence of products of squares and $x = f^\circ \operatorname{len} f$.

We introduce x is a product of squares as a synonym of x is a product of squares.

Let us consider R, f. We say that f is sequence of sums of products of squares if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) $\operatorname{len} f \neq 0$,

(ii) $f^\circ 1$ is a product of squares, and

(iii) for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is a product of squares and $f^\circ(n+1) = f^\circ n + y$.

We introduce f is a sequence of sums of products of squares as a synonym of f is sequence of sums of products of squares.

Let us consider R, x. We say that x is a sum of products of squares if and only if:

(Def. 9) There exists f such that f is a sequence of sums of products of squares and $x = f^\circ \operatorname{len} f$.

We introduce x is a sum of products of squares as a synonym of x is a sum of products of squares.

Let us consider R, f. We say that f is sequence of amalgams of squares if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $\operatorname{len} f \neq 0$, and

(ii) for every n such that $n \neq 0$ and $n \leq \operatorname{len} f$ holds $f^\circ n$ is a product of squares or there exist i, j such that $f^\circ n = f^\circ i \cdot f^\circ j$ and $i \neq 0$ and $i < n$ and $j \neq 0$ and $j < n$.

We introduce f is a sequence of amalgams of squares as a synonym of f is sequence of amalgams of squares.

Let us consider R, x. We say that x is an amalgam of squares if and only if:

(Def. 11) There exists f such that f is a sequence of amalgams of squares and $x = f^\circ \operatorname{len} f$.

We introduce x is an amalgam of squares as a synonym of x is an amalgam of squares.

Let us consider R, f. We say that f is sequence of sums of amalgams of squares if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) $\operatorname{len} f \neq 0$,

(ii) $f^\circ 1$ is an amalgam of squares, and

(iii) for every n such that $n \neq 0$ and $n < \operatorname{len} f$ there exists y such that y is an amalgam of squares and $f^\circ(n+1) = f^\circ n + y$.

We introduce f is a sequence of sums of amalgams of squares as a synonym of f is sequence of sums of amalgams of squares.

Let us consider R, x. We say that x is a sum of amalgams of squares if and only if:

(Def. 13) There exists f such that f is a sequence of sums of amalgams of squares and $x = f^\circ \operatorname{len} f$.

We introduce x is a sum of amalgams of squares as a synonym of x is a sum of amalgams of squares.

Let us consider R, f. We say that f is a generation from squares if and only if the conditions (Def. 14) are satisfied.

(Def. 14)(i) $\operatorname{len} f \neq 0$, and

(ii) for every n such that $n \neq 0$ and $n \leq \operatorname{len} f$ holds $f^\circ n$ is an amalgam of squares or there exist i, j such that $f^\circ n = f^\circ i \cdot f^\circ j$ or $f^\circ n = f^\circ i + f^\circ j$ and $i \neq 0$ and $i < n$ and $j \neq 0$ and $j < n$.

We introduce f is a generation from squares as a synonym of f is a generation from squares.

Let us consider R, x. We say that x is generated from squares if and only if:

(Def. 15) There exists f such that f is a generation from squares and $x = f^\circ \operatorname{len} f$.

We introduce x is generated from squares as a synonym of x is generated from squares.

Next we state several propositions:

(1) Suppose x is a square. Then x is a sum of squares, a product of squares, a sum of products of squares, an amalgam of squares, a sum of amalgams of squares, and generated from squares.

(2) Suppose x is a sum of squares. Then x is a sum of products of squares, a sum of amalgams of squares, and generated from squares.

(3) Suppose x is a product of squares. Then x is a sum of products of squares, an amalgam of squares, a sum of amalgams of squares, and generated from squares.

(4) If x is a sum of products of squares, then x is a sum of amalgams of squares and generated from squares.

(5) If x is an amalgam of squares, then x is a sum of amalgams of squares and generated from squares.

(6) If x is a sum of amalgams of squares, then x is generated from squares.

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