JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
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Real Normed Space

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Summary. We construct a real normed space $\langle V, \|.\|\rangle$, where V is a real vector space and $\|.\|$ is a norm. Auxillary properties of the norm are proved. Next, we introduce a notion of sequence in the real normed space. The basic operations on sequences (addition, subtraction, multiplication by real number) are defined. We study some properties of sequences in the real normed space and the operations on them.

MML Identifier: `NORMSP_1`.

WWW: `http://mizar.org/JFM/Vol2/normsp_1.html`

The articles [11], [5], [14], [2], [12], [6], [1], [3], [4], [15], [9], [7], [8], [10], and [13] provide the notation and terminology for this paper.

We introduce normed structures which are extensions of RLS structure and are systems

$\langle$ a carrier, a zero, an addition, an external multiplication, a norm $\rangle$,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[:\mathbb{R}, \text{the carrier}:]$ into the carrier, and the norm is a function from the carrier into $\mathbb{R}$.

Let us note that there exists a normed structure which is non empty.

In the sequel X denotes a non empty normed structure, a, b denote real numbers, and x denotes a point of X.

Let us consider X, x. The functor $\|x\|$ yielding a real number is defined as follows:

(Def. 1) $\|x\| = (\text{the norm of } X)(x)$.

Let I_1 be a non empty normed structure. We say that I_1 is real normed space-like if and only if:

(Def. 2) For all points x, y of I_1 and for every a holds $\|x\| = 0$ iff $x = 0_{(I_1)}$ and $\|a \cdot x\| = |a| \cdot \|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$.

One can check that there exists a non empty normed structure which is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A real normed space is a real normed space-like real linear space-like Abelian add-associative right zeroed right complementable non empty normed structure.

We adopt the following convention: R_1 denotes a real normed space and x, y, z, g denote points of R_1.

Next we state a number of propositions:

(5)[1] $\|0_{(R_1)}\| = 0$.

(6) $\|-x\| = \|x\|$.

[1] The propositions (1)–(4) have been removed.

(7) $\|x-y\| \leq \|x\| + \|y\|$.

(8) $0 \leq \|x\|$.

(9) $\|a \cdot x + b \cdot y\| \leq |a| \cdot \|x\| + |b| \cdot \|y\|$.

(10) $\|x-y\| = 0$ iff $x = y$.

(11) $\|x-y\| = \|y-x\|$.

(12) $\|x\| - \|y\| \leq \|x-y\|$.

(13) $|\|x\| - \|y\|| \leq \|x-y\|$.

(14) $\|x-z\| \leq \|x-y\| + \|y-z\|$.

(15) If $x \neq y$, then $\|x-y\| \neq 0$.

Let R_1 be a 1-sorted structure. Sequence of R_1 is defined by:

(Def. 3) It is a function from $\mathbb{N}$ into the carrier of R_1.

Let R_1 be a 1-sorted structure. Observe that every sequence of R_1 is function-like and relation-like.

Let R_1 be a non empty 1-sorted structure. We see that the sequence of R_1 is a function from $\mathbb{N}$ into the carrier of R_1.

For simplicity, we follow the rules: S, S_1, S_2 are sequences of R_1, k, n, m are natural numbers, r is a real number, f is a function, and d is a set.

We now state several propositions:

(17)[2] Let R_1 be a non empty 1-sorted structure and x be an element of R_1. Then f is a sequence of R_1 if and only if $\operatorname{dom} f = \mathbb{N}$ and for every d such that $d \in \mathbb{N}$ holds $f(d)$ is an element of R_1.

(19)[3] For every non empty 1-sorted structure R_1 and for every element x of R_1 there exists a sequence S of R_1 such that $\operatorname{rng} S = \{x\}$.

(20) Let R_1 be a non empty 1-sorted structure and S be a sequence of R_1. Given an element x of R_1 such that let given n. Then $S(n) = x$. Then there exists an element x of R_1 such that $\operatorname{rng} S = \{x\}$.

(21) Let R_1 be a non empty 1-sorted structure and S be a sequence of R_1. If there exists an element x of R_1 such that $\operatorname{rng} S = \{x\}$, then for every n holds $S(n) = S(n+1)$.

(22) Let R_1 be a non empty 1-sorted structure and S be a sequence of R_1. If for every n holds $S(n) = S(n+1)$, then for all n, k holds $S(n) = S(n+k)$.

(23) Let R_1 be a non empty 1-sorted structure and S be a sequence of R_1. If for all n, k holds $S(n) = S(n+k)$, then for all n, m holds $S(n) = S(m)$.

(24) Let R_1 be a non empty 1-sorted structure and S be a sequence of R_1. Suppose that for all n, m holds $S(n) = S(m)$. Then there exists an element x of R_1 such that for every n holds $S(n) = x$.

(25) There exists S such that $\operatorname{rng} S = \{0_{(R_1)}\}$.

Let R_1 be a non empty 1-sorted structure and let S be a sequence of R_1. Let us observe that S is constant if and only if:

(Def. 4) There exists an element x of R_1 such that for every n holds $S(n) = x$.

[2] The proposition (16) has been removed.

[3] The proposition (18) has been removed.

The following proposition is true

(27)[4] Let R_1 be a non empty 1-sorted structure and S be a sequence of R_1. Then S is constant if and only if there exists an element x of R_1 such that $\operatorname{rng} S = \{x\}$.

Let R_1 be a non empty 1-sorted structure, let S be a sequence of R_1, and let us consider n. Then $S(n)$ is an element of R_1.

In this article we present several logical schemes. The scheme *ExRNSSeq* deals with a real normed space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExRLSSeq* deals with a real linear space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Let R_1 be a real linear space and let S_1, S_2 be sequences of R_1. The functor $S_1 + S_2$ yielding a sequence of R_1 is defined as follows:

(Def. 5) For every n holds $(S_1 + S_2)(n) = S_1(n) + S_2(n)$.

Let R_1 be a real linear space and let S_1, S_2 be sequences of R_1. The functor $S_1 - S_2$ yields a sequence of R_1 and is defined as follows:

(Def. 6) For every n holds $(S_1 - S_2)(n) = S_1(n) - S_2(n)$.

Let R_1 be a real linear space, let S be a sequence of R_1, and let x be an element of R_1. The functor $S - x$ yields a sequence of R_1 and is defined by:

(Def. 7) For every n holds $(S - x)(n) = S(n) - x$.

Let R_1 be a real linear space, let S be a sequence of R_1, and let us consider a. The functor $a \cdot S$ yielding a sequence of R_1 is defined as follows:

(Def. 8) For every n holds $(a \cdot S)(n) = a \cdot S(n)$.

Let us consider R_1 and let us consider S. We say that S is convergent if and only if:

(Def. 9) There exists g such that for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - g\| < r$.

One can prove the following propositions:

(34)[5] If S_1 is convergent and S_2 is convergent, then $S_1 + S_2$ is convergent.

(35) If S_1 is convergent and S_2 is convergent, then $S_1 - S_2$ is convergent.

(36) If S is convergent, then $S - x$ is convergent.

(37) If S is convergent, then $a \cdot S$ is convergent.

Let us consider R_1 and let us consider S. The functor $\|S\|$ yields a sequence of real numbers and is defined as follows:

(Def. 10) For every n holds $\|S\|(n) = \|S(n)\|$.

Next we state the proposition

(39)[6] If S is convergent, then $\|S\|$ is convergent.

[4] The proposition (26) has been removed.
[5] The propositions (28)–(33) have been removed.
[6] The proposition (38) has been removed.

Let us consider R_1 and let us consider S. Let us assume that S is convergent. The functor $\lim S$ yielding a point of R_1 is defined by:

(Def. 11) For every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - \lim S\| < r$.

One can prove the following propositions:

(41)[7] If S is convergent and $\lim S = g$, then $\|S - g\|$ is convergent and $\lim \|S - g\| = 0$.

(42) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 + S_2) = \lim S_1 + \lim S_2$.

(43) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 - S_2) = \lim S_1 - \lim S_2$.

(44) If S is convergent, then $\lim(S - x) = \lim S - x$.

(45) If S is convergent, then $\lim(a \cdot S) = a \cdot \lim S$.

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Received September 20, 1990

Published January 2, 2004

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[7] The proposition (40) has been removed.