JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

# On the Trivial Many Sorted Algebras and Many Sorted Congruences

Artur Korniłowicz
Institute of Mathematics
Warsaw University
Białystok

**Summary.** This paper contains properties of many sorted functions between two many sorted sets. Other theorems describe trivial many sorted algebras. In the last section there are theorems about many sorted congruences, which are defined on many sorted algebras. I have also proved facts about natural epimorphism.

MML Identifier: `MSUALG_9`.

WWW: `http://mizar.org/JFM/Vol8/msualg_9.html`

The articles [26], [8], [31], [30], [32], [34], [23], [33], [6], [25], [7], [3], [9], [27], [10], [1], [2], [4], [28], [29], [5], [11], [17], [24], [19], [21], [22], [18], [13], [14], [12], [16], [15], and [20] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

In this paper $a$, $I$ denote sets and $S$ denotes a non empty non void many sorted signature.

The scheme *MSSExD* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a many sorted set $f$ indexed by $\mathcal{A}$ such that for every element $i$ of $\mathcal{A}$ holds $\mathcal{P}[i, f(i)]$

provided the following condition is met:

- For every element $i$ of $\mathcal{A}$ there exists a set $j$ such that $\mathcal{P}[i, j]$.

Let $I$ be a set and let $M$ be a many sorted set indexed by $I$. One can check that there exists an element of $\mathrm{Bool}(M)$ which is locally-finite.

Let $I$ be a set and let $M$ be a non-empty many sorted set indexed by $I$. Note that there exists a many sorted subset indexed by $M$ which is non-empty and locally-finite.

Let $S$ be a non empty non void many sorted signature, let $A$ be a non-empty algebra over $S$, and let $o$ be an operation symbol of $S$. Note that every element of $\mathrm{Args}(o,A)$ is finite sequence-like.

Let $S$ be a non void non empty many sorted signature, let $I$ be a set, let $s$ be a sort symbol of $S$, and let $F$ be an algebra family of $I$ over $S$. Observe that every element of $(\mathrm{SORTS}(F))(s)$ is function-like and relation-like.

Let $S$ be a non void non empty many sorted signature and let $X$ be a non-empty many sorted set indexed by the carrier of $S$. One can verify that $\mathrm{FreeGenerator}(X)$ is free and non-empty.

Let $S$ be a non void non empty many sorted signature and let $X$ be a non-empty many sorted set indexed by the carrier of $S$. Observe that $\mathrm{Free}(X)$ is free.

Let $S$ be a non empty non void many sorted signature and let $A$, $B$ be non-empty algebras over $S$. Note that $[:A, B:]$ is non-empty.

The following propositions are true:

(1) For all sets $X$, $Y$ and for every function $f$ such that $a \in \operatorname{dom} f$ and $f(a) \in [:X, Y:]$ holds $f(a) = \langle \mathrm{pr1}(f)(a), \mathrm{pr2}(f)(a) \rangle$.

(2) For every non empty set $X$ and for every set $Y$ and for every function $f$ from $X$ into $\{Y\}$ holds $\operatorname{rng} f = \{Y\}$.

(3) For every non empty finite set $A$ there exists a function $f$ from $\mathbb{N}$ into $A$ such that $\operatorname{rng} f = A$.

(4) $\mathrm{Classes}(\nabla_I) \subseteq \{I\}$.

(5) For every non empty set $I$ holds $\mathrm{Classes}(\nabla_I) = \{I\}$.

(6) There exists a many sorted set $A$ indexed by $I$ such that $\{A\} = I \longmapsto \{a\}$.

(7) For every many sorted set $A$ indexed by $I$ there exists a non-empty many sorted set $B$ indexed by $I$ such that $A \subseteq B$.

(8) Let $M$ be a non-empty many sorted set indexed by $I$ and $B$ be a locally-finite many sorted subset indexed by $M$. Then there exists a non-empty locally-finite many sorted subset $C$ indexed by $M$ such that $B \subseteq C$.

(9) For all many sorted sets $A$, $B$ indexed by $I$ and for all many sorted functions $F$, $G$ from $A$ into $\{B\}$ holds $F = G$.

(10) For every non-empty many sorted set $A$ indexed by $I$ and for every many sorted set $B$ indexed by $I$ holds every many sorted function from $A$ into $\{B\}$ is onto.

(11) Let $A$ be a many sorted set indexed by $I$ and $B$ be a non-empty many sorted set indexed by $I$. Then every many sorted function from $\{A\}$ into $B$ is "1-1".

(12) For every non-empty many sorted set $X$ indexed by the carrier of $S$ holds $\mathrm{Reverse}(X)$ is "1-1".

(13) For every non-empty locally-finite many sorted set $A$ indexed by $I$ holds there exists a many sorted function from $I \longmapsto \mathbb{N}$ into $A$ which is onto.

(14) Let $S$ be a non empty many sorted signature, $A$ be a non-empty algebra over $S$, and $f$, $g$ be elements of $\prod$(the sorts of $A$). Suppose that for every set $i$ holds (proj(the sorts of $A$, $i$))$(f)$ = (proj(the sorts of $A$, $i$))$(g)$. Then $f = g$.

(15) Let $I$ be a non empty set, $s$ be an element of $S$, $A$ be an algebra family of $I$ over $S$, and $f$, $g$ be elements of $\prod \mathrm{Carrier}(A,s)$. If for every element $a$ of $I$ holds $(\mathrm{proj}(\mathrm{Carrier}(A,s),a))(f) = (\mathrm{proj}(\mathrm{Carrier}(A,s),a))(g)$, then $f = g$.

(16) Let $A$, $B$ be non-empty algebras over $S$, $C$ be a non-empty subalgebra of $A$, and $h_1$ be a many sorted function from $B$ into $C$. Suppose $h_1$ is a homomorphism of $B$ into $C$. Let $h_2$ be a many sorted function from $B$ into $A$. If $h_1 = h_2$, then $h_2$ is a homomorphism of $B$ into $A$.

(17) Let $A$, $B$ be non-empty algebras over $S$ and $F$ be a many sorted function from $A$ into $B$. If $F$ is a monomorphism of $A$ into $B$, then $A$ and $\operatorname{Im} F$ are isomorphic.

(18) Let $A$, $B$ be non-empty algebras over $S$ and $F$ be a many sorted function from $A$ into $B$. Suppose $F$ is onto. Let $o$ be an operation symbol of $S$ and $x$ be an element of $\mathrm{Args}(o,B)$. Then there exists an element $y$ of $\mathrm{Args}(o,A)$ such that $F\#y = x$.

(19) Let $A$ be a non-empty algebra over $S$, $o$ be an operation symbol of $S$, and $x$ be an element of $\mathrm{Args}(o,A)$. Then $(\mathrm{Den}(o,A))(x) \in$ (the sorts of $A$)(the result sort of $o$).

(20) Let $A$, $B$, $C$ be non-empty algebras over $S$, $F_1$ be a many sorted function from $A$ into $B$, and $F_2$ be a many sorted function from $A$ into $C$. Suppose $F_1$ is an epimorphism of $A$ onto $B$ and $F_2$ is a homomorphism of $A$ into $C$. Let $G$ be a many sorted function from $B$ into $C$. If $G \circ F_1 = F_2$, then $G$ is a homomorphism of $B$ into $C$.

In the sequel $A$, $M$ denote many sorted sets indexed by $I$ and $B$, $C$ denote non-empty many sorted sets indexed by $I$.

Let $I$ be a set, let $A$ be a many sorted set indexed by $I$, let $B$, $C$ be non-empty many sorted sets indexed by $I$, and let $F$ be a many sorted function from $A$ into $[\![B,C]\!]$. The functor $\mathrm{Mpr1}(F)$ yielding a many sorted function from $A$ into $B$ is defined by:

(Def. 1) For every set $i$ such that $i \in I$ holds $(\mathrm{Mpr1}(F))(i) = \mathrm{pr1}(F(i))$.

The functor $\mathrm{Mpr2}(F)$ yielding a many sorted function from $A$ into $C$ is defined as follows:

(Def. 2) For every set $i$ such that $i \in I$ holds $(\mathrm{Mpr2}(F))(i) = \mathrm{pr2}(F(i))$.

We now state four propositions:

(21) For every many sorted function $F$ from $A$ into $[\![I \longmapsto \{a\}, I \longmapsto \{a\}]\!]$ holds $\mathrm{Mpr1}(F) = \mathrm{Mpr2}(F)$.

(22) For every many sorted function $F$ from $A$ into $[\![B,C]\!]$ such that $F$ is onto holds $\mathrm{Mpr1}(F)$ is onto.

(23) For every many sorted function $F$ from $A$ into $[\![B,C]\!]$ such that $F$ is onto holds $\mathrm{Mpr2}(F)$ is onto.

(24) Let $F$ be a many sorted function from $A$ into $[\![B,C]\!]$. If $M \in \mathrm{dom}_{\kappa} F(\kappa)$, then for every set $i$ such that $i \in I$ holds $(F \looparrowleft M)(i) = \langle((\mathrm{Mpr1}(F)) \looparrowleft M)(i), ((\mathrm{Mpr2}(F)) \looparrowleft M)(i)\rangle$.

## 2. On the Trivial Many Sorted Algebras

Let $S$ be a non empty many sorted signature. Observe that the sorts of the trivial algebra of $S$ is locally-finite and non-empty.

Let $S$ be a non empty many sorted signature. Observe that the trivial algebra of $S$ is locally-finite and non-empty.

We now state three propositions:

(25) Let $A$ be a non-empty algebra over $S$, $F$ be a many sorted function from $A$ into the trivial algebra of $S$, $o$ be an operation symbol of $S$, and $x$ be an element of $\mathrm{Args}(o,A)$. Then $F(\text{the result sort of } o)((\mathrm{Den}(o,A))(x)) = 0$ and $(\mathrm{Den}(o, \text{the trivial algebra of } S))(F\#x) = 0$.

(26) For every non-empty algebra $A$ over $S$ holds every many sorted function from $A$ into the trivial algebra of $S$ is an epimorphism of $A$ onto the trivial algebra of $S$.

(27) Let $A$ be an algebra over $S$. Given a many sorted set $X$ indexed by the carrier of $S$ such that the sorts of $A = \{X\}$. Then $A$ and the trivial algebra of $S$ are isomorphic.

## 3. On the Many Sorted Congruences

The following propositions are true:

(28) Let $A$ be a non-empty algebra over $S$. Then every congruence of $A$ is a many sorted subset indexed by $[\![\text{the sorts of } A, \text{the sorts of } A]\!]$.

(29) Let $A$ be a non-empty algebra over $S$, $R$ be a subset of $\mathrm{CongrLatt}(A)$, and $F$ be a family of many sorted subsets indexed by $[\![\text{the sorts of } A, \text{the sorts of } A]\!]$. If $R = F$, then $\bigcap|{:}F{:}|$ is a congruence of $A$.

(30) Let $A$ be a non-empty algebra over $S$ and $C$ be a congruence of $A$. Suppose $C = [\![\text{the sorts of } A, \text{the sorts of } A]\!]$. Then the sorts of $A/C = \{\text{the sorts of } A\}$.

(31) Let $A$, $B$ be non-empty algebras over $S$ and $F$ be a many sorted function from $A$ into $B$. If $F$ is a homomorphism of $A$ into $B$, then $\mathrm{MSHomQuot}(F) \circ \mathrm{MSNatHom}(A, \mathrm{Congruence}(F)) = F$.

(32) Let $A$ be a non-empty algebra over $S$, $C$ be a congruence of $A$, $s$ be a sort symbol of $S$, and $a$ be an element of (the sorts of $A/C)(s)$. Then there exists an element $x$ of (the sorts of $A)(s)$ such that $a = [x]_C$.

(33) Let $A$ be an algebra over $S$ and $C_1$, $C_2$ be equivalence many sorted relations indexed by $A$. Suppose $C_1 \subseteq C_2$. Let $i$ be an element of $S$ and $x$, $y$ be elements of (the sorts of $A)(i)$. If $\langle x, y\rangle \in C_1(i)$, then $[x]_{(C_1)} \subseteq [y]_{(C_2)}$ and if $A$ is non-empty, then $[y]_{(C_1)} \subseteq [x]_{(C_2)}$.

(34) Let $A$ be a non-empty algebra over $S$, $C$ be a congruence of $A$, $s$ be a sort symbol of $S$, and $x$, $y$ be elements of (the sorts of $A)(s)$. Then $(\mathrm{MSNatHom}(A,C))(s)(x) = (\mathrm{MSNatHom}(A,C))(s)(y)$ if and only if $\langle x, y\rangle \in C(s)$.

(35) Let $A$ be a non-empty algebra over $S$, $C_1$, $C_2$ be congruences of $A$, and $G$ be a many sorted function from $A/C_1$ into $A/C_2$. Suppose that for every element $i$ of $S$ and for every element $x$ of (the sorts of $A/C_1)(i)$ and for every element $x_1$ of (the sorts of $A)(i)$ such that $x = [x_1]_{(C_1)}$ holds $G(i)(x) = [x_1]_{(C_2)}$. Then $G \circ \mathrm{MSNatHom}(A,C_1) = \mathrm{MSNatHom}(A,C_2)$.

(36) Let $A$ be a non-empty algebra over $S$, $C_1$, $C_2$ be congruences of $A$, and $G$ be a many sorted function from $A/C_1$ into $A/C_2$. Suppose that for every element $i$ of $S$ and for every element $x$ of (the sorts of $A/C_1)(i)$ and for every element $x_1$ of (the sorts of $A)(i)$ such that $x = [x_1]_{(C_1)}$ holds $G(i)(x) = [x_1]_{(C_2)}$. Then $G$ is an epimorphism of $A/C_1$ onto $A/C_2$.

(37) Let $A$, $B$ be non-empty algebras over $S$ and $F$ be a many sorted function from $A$ into $B$. Suppose $F$ is a homomorphism of $A$ into $B$. Let $C_1$ be a congruence of $A$. Suppose $C_1 \subseteq \mathrm{Congruence}(F)$. Then there exists a many sorted function $H$ from $A/C_1$ into $B$ such that $H$ is a homomorphism of $A/C_1$ into $B$ and $F = H \circ \mathrm{MSNatHom}(A,C_1)$.

## REFERENCES


[1] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[2] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/trees_4.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/dtconstr.html`.

[5] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_2.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[9] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lang1.html`.

[10] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[11] Mariusz Giero. More on products of many sorted algebras. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/pralg_3.html`.

[12] Artur Korniłowicz. Certain facts about families of subsets of many sorted sets. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/mssubfam.html`.

[13] Artur Korniłowicz. Extensions of mappings on generator set. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/extens_1.html`.

[14] Artur Korniłowicz. Some basic properties of many sorted sets. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/pzfmisc1.html`.

[15] Artur Korniłowicz. On the closure operator and the closure system of many sorted sets. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/closure2.html`.

[16] Artur Korniłowicz. On the many sorted closure operator and the many sorted closure system. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/closure1.html.

[17] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[18] Małgorzata Korolkiewicz. Many sorted quotient algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_4.html.

[19] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pralg_2.html.

[20] Robert Milewski. Lattice of congruences in many sorted algebra. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/msualg_5.html.

[21] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pre_circ.html.

[22] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree2.html.

[23] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[24] Beata Perkowska. Free many sorted universal algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree.html.

[25] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/eqrel_1.html.

[26] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[27] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[28] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[29] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[30] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[31] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[32] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[33] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[34] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

*Received June 11, 1996*

*Published January 2, 2004*