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More on the Lattice of Congruences in Many Sorted Algebra

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The articles [16], [8], [19], [14], [20], [21], [1], [5], [7], [6], [9], [4], [2], [15], [22], [3], [17], [11], [18], [10], [12], and [13] provide the notation and terminology for this paper.

1. More on the Lattice of Equivalence Relations

For simplicity, we adopt the following convention: I denotes a non empty set, M denotes a many sorted set indexed by I, Y, x, y denote sets, k denotes a natural number, p denotes a finite sequence, S denotes a non void non empty many sorted signature, and A denotes a non-empty algebra over S.

The following proposition is true

(1) For every natural number n and for every finite sequence p holds $1 \leq n$ and $n < \operatorname{len} p$ iff $n \in \operatorname{dom} p$ and $n+1 \in \operatorname{dom} p$.

The scheme *NonUniqSeqEx* deals with a natural number $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists p such that $\operatorname{dom} p = \operatorname{Seg} \mathcal{A}$ and for every k such that $k \in \operatorname{Seg} \mathcal{A}$ holds $\mathcal{P}[k, p(k)]$

provided the following condition is satisfied:

- For every k such that $k \in \operatorname{Seg} \mathcal{A}$ there exists x such that $\mathcal{P}[k,x]$.

Next we state the proposition

(2) Let a, b be elements of $\operatorname{EqRelLatt}(Y)$ and A, B be equivalence relations of Y. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

Let us consider Y. Note that $\operatorname{EqRelLatt}(Y)$ is bounded.

Next we state three propositions:

(3) $\perp_{\operatorname{EqRelLatt}(Y)} = \operatorname{id}_Y$.

(4) $\top_{\operatorname{EqRelLatt}(Y)} = \nabla_Y$.

(5) $\operatorname{EqRelLatt}(Y)$ is complete.

Let us consider Y. Note that $\operatorname{EqRelLatt}(Y)$ is complete.

Next we state several propositions:

(6) For every set Y and for every subset X of $\mathrm{EqRelLatt}(Y)$ holds $\bigcup X$ is a binary relation on Y.

(7) For every set Y and for every subset X of $\mathrm{EqRelLatt}(Y)$ holds $\bigcup X \subseteq \bigsqcup X$.

(8) For every set Y and for every subset X of $\mathrm{EqRelLatt}(Y)$ and for every binary relation R on Y such that $R = \bigcup X$ holds $\bigsqcup X = \mathrm{EqCl}(R)$.

(9) For every set Y and for every subset X of $\mathrm{EqRelLatt}(Y)$ and for every binary relation R such that $R = \bigcup X$ holds $R = R^{\smile}$.

(10) Let Y be a set and X be a subset of $\mathrm{EqRelLatt}(Y)$. Suppose $x \in Y$ and $y \in Y$. Then $\langle x, y\rangle \in \bigsqcup X$ if and only if there exists a finite sequence f such that $1 \leq \mathrm{len}\, f$ and $x = f(1)$ and $y = f(\mathrm{len}\, f)$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, f$ holds $\langle f(i), f(i+1)\rangle \in \bigcup X$.

2. Lattice of Congruences in Many Sorted Algebra as Sublattice of Lattice of Many Sorted Equivalence Relations Inherited Sup's and Inf's

One can prove the following proposition

(11) For every subset B of $\mathrm{CongrLatt}(A)$ holds $\bigsqcap_{\mathrm{EqRelLatt}(\text{the sorts of } A)} B$ is a congruence of A.

Let us consider S, A and let E be an element of EqRelLatt(the sorts of A). The functor $\mathrm{CongrCl}(E)$ yields a congruence of A and is defined by:

(Def. 1) $\mathrm{CongrCl}(E) = \bigsqcap_{\mathrm{EqRelLatt}(\text{the sorts of } A)} \{x; x$ ranges over elements of EqRelLatt(the sorts of A): x is a congruence of $A \wedge E \sqsubseteq x\}$.

Let us consider S, A and let X be a subset of EqRelLatt(the sorts of A). The functor $\mathrm{CongrCl}(X)$ yielding a congruence of A is defined as follows:

(Def. 2) $\mathrm{CongrCl}(X) = \bigsqcap_{\mathrm{EqRelLatt}(\text{the sorts of } A)} \{x; x$ ranges over elements of EqRelLatt(the sorts of A): x is a congruence of $A \wedge X \sqsubseteq x\}$.

One can prove the following propositions:

(12) For every element C of EqRelLatt(the sorts of A) such that C is a congruence of A holds $\mathrm{CongrCl}(C) = C$.

(13) For every subset X of EqRelLatt(the sorts of A) holds $\mathrm{CongrCl}(\bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} X) = \mathrm{CongrCl}(X)$.

(14) Let B_1, B_2 be subsets of $\mathrm{CongrLatt}(A)$ and C_1, C_2 be congruences of A. Suppose $C_1 = \bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} B_1$ and $C_2 = \bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} B_2$. Then $C_1 \sqcup C_2 = \bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} (B_1 \cup B_2)$.

(15) Let X be a subset of $\mathrm{CongrLatt}(A)$. Then $\bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} X = \bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} \{\bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} X_0; X_0$ ranges over subsets of EqRelLatt(the sorts of A): X_0 is a finite subset of $X\}$.

(16) Let i be an element of I and e be an equivalence relation of $M(i)$. Then there exists an equivalence relation E of M such that $E(i) = e$ and for every element j of I such that $j \neq i$ holds $E(j) = \nabla_{M(j)}$.

Let I be a non empty set, let M be a many sorted set indexed by I, let i be an element of I, and let X be a subset of $\mathrm{EqRelLatt}(M)$. Then $\pi_i X$ is a subset of $\mathrm{EqRelLatt}(M(i))$ and it can be characterized by the condition:

(Def. 3) $x \in \pi_i X$ iff there exists an equivalence relation E_1 of M such that $x = E_1(i)$ and $E_1 \in X$.

We introduce $\mathrm{EqRelSet}(X, i)$ as a synonym of $\pi_i X$.

The following four propositions are true:

(17) Let i be an element of S, X be a subset of EqRelLatt(the sorts of A), and B be an equivalence relation of the sorts of A. If $B = \bigsqcup X$, then $B(i) = \bigsqcup_{\mathrm{EqRelLatt}((\text{the sorts of } A)(i))} \mathrm{EqRelSet}(X, i)$.

(18) For every subset X of CongrLatt(A) holds $\bigsqcup_{\mathrm{EqRelLatt}(\text{the sorts of } A)} X$ is a congruence of A.

(19) CongrLatt(A) is $\sqcap$-inheriting.

(20) CongrLatt(A) is $\bigsqcup$-inheriting.

Let us consider S, A. Note that CongrLatt(A) is $\sqcap$-inheriting and $\bigsqcup$-inheriting.

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