JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Translations, Endomorphisms, and Stable Equational Theories

Grzegorz Bancerek
Institute of Mathematics
Polish Academy of Sciences

Summary. Equational theories of an algebra, i.e. the equivalence relation closed under translations and endomorphisms, are formalized. The correspondence between equational theories and term rewriting systems is discussed in the paper. We get as the main result that any pair of elements of an algebra belongs to the equational theory generated by a set A of axioms iff the elements are convertible w.r.t. term rewriting reduction determined by A.

The theory is developed according to [19].

MML Identifier: MSUALG_6.

WWW: http://mizar.org/JFM/Vol8/msualg_6.html

The articles [14], [9], [18], [1], [20], [22], [21], [3], [6], [8], [7], [4], [2], [17], [15], [12], [16], [10], [11], [13], and [5] provide the notation and terminology for this paper.

1. ENDOMORPHISMS AND TRANSLATIONS

Let S be a non empty many sorted signature, let A be an algebra over S, and let s be a sort symbol of S. An element of A, s is an element of (the sorts of $A)(s)$.

Let I be a set, let A be a many sorted set indexed by I, and let h_1, h_2 be many sorted functions from A into A. Then $h_2 \circ h_1$ is a many sorted function from A into A.

One can prove the following propositions:

(1) Let S be a non empty non void many sorted signature, A be an algebra over S, o be an operation symbol of S, and a be a set. If $a \in \mathrm{Args}(o,A)$, then a is a function.

(2) Let S be a non empty non void many sorted signature, A be an algebra over S, o be an operation symbol of S, and a be a function. Suppose $a \in \mathrm{Args}(o,A)$. Then $\mathrm{dom}\, a = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every set i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $a(i) \in$ (the sorts of $A)(\mathrm{Arity}(o)_i)$.

Let S be a non empty non void many sorted signature and let A be an algebra over S. We say that A is feasible if and only if:

(Def. 1) For every operation symbol o of S such that $\mathrm{Args}(o,A) \neq \emptyset$ holds $\mathrm{Result}(o,A) \neq \emptyset$.

Next we state the proposition

(3) Let S be a non empty non void many sorted signature, o be an operation symbol of S, and A be an algebra over S. Then $\mathrm{Args}(o,A) \neq \emptyset$ if and only if for every natural number i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds (the sorts of $A)(\mathrm{Arity}(o)_i) \neq \emptyset$.

Let S be a non empty non void many sorted signature. One can verify that every algebra over S which is non-empty is also feasible.

Let S be a non empty non void many sorted signature. One can check that there exists an algebra over S which is non-empty.

Let S be a non empty non void many sorted signature and let A be an algebra over S. A many sorted function from A into A is said to be an endomorphism of A if:

(Def. 2) It is a homomorphism of A into A.

In the sequel S is a non empty non void many sorted signature and A is an algebra over S.

The following propositions are true:

(4) $\mathrm{id}_{\text{the sorts of } A}$ is an endomorphism of A.

(5) Let h_1, h_2 be many sorted functions from A into A, o be an operation symbol of S, and a be an element of $\mathrm{Args}(o,A)$. If $a \in \mathrm{Args}(o,A)$, then $h_2\#(h_1\#a) = (h_2 \circ h_1)\#a$.

(6) For all endomorphisms h_1, h_2 of A holds $h_2 \circ h_1$ is an endomorphism of A.

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let h_1, h_2 be endomorphisms of A. Then $h_2 \circ h_1$ is an endomorphism of A.

Let S be a non empty non void many sorted signature. The functor $\mathrm{TranslRel}(S)$ is a binary relation on the carrier of S and is defined by the condition (Def. 3).

(Def. 3) Let s_1, s_2 be sort symbols of S. Then $\langle s_1, s_2 \rangle \in \mathrm{TranslRel}(S)$ if and only if there exists an operation symbol o of S such that the result sort of $o = s_2$ and there exists a natural number i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ and $\mathrm{Arity}(o)_i = s_1$.

One can prove the following three propositions:

(7) Let S be a non empty non void many sorted signature, o be an operation symbol of S, A be an algebra over S, and a be a function. Suppose $a \in \mathrm{Args}(o,A)$. Let i be a natural number and x be an element of A, $\mathrm{Arity}(o)_i$. Then $a +\cdot (i,x) \in \mathrm{Args}(o,A)$.

(8) Let A_1, A_2 be algebras over S, h be a many sorted function from A_1 into A_2, and o be an operation symbol of S. Suppose $\mathrm{Args}(o,A_1) \neq \emptyset$ and $\mathrm{Args}(o,A_2) \neq \emptyset$. Let i be a natural number. Suppose $i \in \mathrm{dom}\,\mathrm{Arity}(o)$. Let x be an element of A_1, $\mathrm{Arity}(o)_i$. Then $h(\mathrm{Arity}(o)_i)(x) \in$ (the sorts of A_2)$(\mathrm{Arity}(o)_i)$.

(9) Let S be a non empty non void many sorted signature, o be an operation symbol of S, and i be a natural number. Suppose $i \in \mathrm{dom}\,\mathrm{Arity}(o)$. Let A_1, A_2 be algebras over S, h be a many sorted function from A_1 into A_2, and a, b be elements of $\mathrm{Args}(o,A_1)$. Suppose $a \in \mathrm{Args}(o,A_1)$ and $h\#a \in \mathrm{Args}(o,A_2)$. Let f, g_1, g_2 be functions. Suppose $f = a$ and $g_1 = h\#a$ and $g_2 = h\#b$. Let x be an element of A_1, $\mathrm{Arity}(o)_i$. If $b = f +\cdot (i,x)$, then $g_2(i) = h(\mathrm{Arity}(o)_i)(x)$ and $h\#b = g_1 +\cdot (i, g_2(i))$.

Let S be a non empty non void many sorted signature, let o be an operation symbol of S, let i be a natural number, let A be an algebra over S, and let a be a function. The functor $o_i^A(a,-)$ yielding a function is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{dom}(o_i^A(a,-)) =$ (the sorts of A)$(\mathrm{Arity}(o)_i)$, and

(ii) for every set x such that $x \in$ (the sorts of A)$(\mathrm{Arity}(o)_i)$ holds $o_i^A(a,-)(x) = (\mathrm{Den}(o, A))(a +\cdot (i,x))$.

One can prove the following proposition

(10) Let S be a non empty non void many sorted signature, o be an operation symbol of S, and i be a natural number. Suppose $i \in \mathrm{dom}\,\mathrm{Arity}(o)$. Let A be a feasible algebra over S and a be a function. Suppose $a \in \mathrm{Args}(o,A)$. Then $o_i^A(a,-)$ is a function from (the sorts of A)$(\mathrm{Arity}(o)_i)$ into (the sorts of A)(the result sort of o).

Let S be a non empty non void many sorted signature, let s_1, s_2 be sort symbols of S, let A be an algebra over S, and let f be a function. We say that f is an elementary translation in A from s_1 into s_2 if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists an operation symbol o of S such that

(i) the result sort of $o = s_2$, and

(ii) there exists a natural number i such that $i \in \operatorname{dom} \operatorname{Arity}(o)$ and $\operatorname{Arity}(o)_i = s_1$ and there exists a function a such that $a \in \operatorname{Args}(o, A)$ and $f = o_i^A(a, -)$.

One can prove the following four propositions:

(11) Let S be a non empty non void many sorted signature, s_1, s_2 be sort symbols of S, A be a feasible algebra over S, and f be a function. Suppose f is an elementary translation in A from s_1 into s_2. Then

(i) f is a function from (the sorts of $A)(s_1)$ into (the sorts of $A)(s_2)$,

(ii) (the sorts of $A)(s_1) \neq \emptyset$, and

(iii) (the sorts of $A)(s_2) \neq \emptyset$.

(12) Let S be a non empty non void many sorted signature, s_1, s_2 be sort symbols of S, A be an algebra over S, and f be a function. If f is an elementary translation in A from s_1 into s_2, then $\langle s_1, s_2 \rangle \in \operatorname{TranslRel}(S)$.

(13) Let S be a non empty non void many sorted signature, s_1, s_2 be sort symbols of S, and A be a non-empty algebra over S. If $\langle s_1, s_2 \rangle \in \operatorname{TranslRel}(S)$, then there exists a function which is an elementary translation in A from s_1 into s_2.

(14) Let S be a non empty non void many sorted signature, A be a feasible algebra over S, and s_1, s_2 be sort symbols of S. Suppose $\operatorname{TranslRel}(S)$ reduces s_1 to s_2. Let q be a reduction sequence w.r.t. $\operatorname{TranslRel}(S)$ and p be a function yielding finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p + 1$,

(ii) $s_1 = q(1)$,

(iii) $s_2 = q(\operatorname{len} q)$, and

(iv) for every natural number i and for every function f and for all sort symbols s_1, s_2 of S such that $i \in \operatorname{dom} p$ and $f = p(i)$ and $s_1 = q(i)$ and $s_2 = q(i+1)$ holds f is an elementary translation in A from s_1 into s_2.

Then

(v) $\operatorname{compose}_{(\text{the sorts of } A)(s_1)} p$ is a function from (the sorts of $A)(s_1)$ into (the sorts of $A)(s_2)$, and

(vi) if $p \neq \emptyset$, then (the sorts of $A)(s_1) \neq \emptyset$ and (the sorts of $A)(s_2) \neq \emptyset$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let s_1, s_2 be sort symbols of S. Let us assume that $\operatorname{TranslRel}(S)$ reduces s_1 to s_2. A function from (the sorts of $A)(s_1)$ into (the sorts of $A)(s_2)$ is said to be a translation in A from s_1 into s_2 if it satisfies the condition (Def. 6).

(Def. 6) There exists a reduction sequence q w.r.t. $\operatorname{TranslRel}(S)$ and there exists a function yielding finite sequence p such that

(i) $\text{it} = \operatorname{compose}_{(\text{the sorts of } A)(s_1)} p$,

(ii) $\operatorname{len} q = \operatorname{len} p + 1$,

(iii) $s_1 = q(1)$,

(iv) $s_2 = q(\operatorname{len} q)$, and

(v) for every natural number i and for every function f and for all sort symbols s_1, s_2 of S such that $i \in \operatorname{dom} p$ and $f = p(i)$ and $s_1 = q(i)$ and $s_2 = q(i+1)$ holds f is an elementary translation in A from s_1 into s_2.

We now state the proposition

(15) Let S be a non empty non void many sorted signature, A be a non-empty algebra over S, and s_1, s_2 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let q be a reduction sequence w.r.t. TranslRel(S) and p be a function yielding finite sequence. Suppose that

(i) $\operatorname{len} q = \operatorname{len} p + 1$,

(ii) $s_1 = q(1)$,

(iii) $s_2 = q(\operatorname{len} q)$, and

(iv) for every natural number i and for every function f and for all sort symbols s_1, s_2 of S such that $i \in \operatorname{dom} p$ and $f = p(i)$ and $s_1 = q(i)$ and $s_2 = q(i+1)$ holds f is an elementary translation in A from s_1 into s_2.

Then $\mathrm{compose}_{(\text{the sorts of } A)(s_1)}\, p$ is a translation in A from s_1 into s_2.

In the sequel A is a non-empty algebra over S.

We now state several propositions:

(16) For every sort symbol s of S holds $\mathrm{id}_{(\text{the sorts of } A)(s)}$ is a translation in A from s into s.

(17) Let s_1, s_2 be sort symbols of S and f be a function. Suppose f is an elementary translation in A from s_1 into s_2. Then TranslRel(S) reduces s_1 to s_2 and f is a translation in A from s_1 into s_2.

(18) Let s_1, s_2, s_3 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2 and TranslRel(S) reduces s_2 to s_3. Let t_1 be a translation in A from s_1 into s_2 and t_2 be a translation in A from s_2 into s_3. Then $t_2 \cdot t_1$ is a translation in A from s_1 into s_3.

(19) Let s_1, s_2, s_3 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let t be a translation in A from s_1 into s_2 and f be a function. Suppose f is an elementary translation in A from s_2 into s_3. Then $f \cdot t$ is a translation in A from s_1 into s_3.

(20) Let s_1, s_2, s_3 be sort symbols of S. Suppose TranslRel(S) reduces s_2 to s_3. Let f be a function. Suppose f is an elementary translation in A from s_1 into s_2. Let t be a translation in A from s_2 into s_3. Then $t \cdot f$ is a translation in A from s_1 into s_3.

The scheme *TranslationInd* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

Let s_1, s_2 be sort symbols of $\mathcal{A}$. Suppose TranslRel($\mathcal{A}$) reduces s_1 to s_2. Let t be a translation in $\mathcal{B}$ from s_1 into s_2. Then $\mathcal{P}[t, s_1, s_2]$

provided the following conditions are satisfied:

- For every sort symbol s of $\mathcal{A}$ holds $\mathcal{P}[\mathrm{id}_{(\text{the sorts of } \mathcal{B})(s)}, s, s]$, and
- Let s_1, s_2, s_3 be sort symbols of $\mathcal{A}$. Suppose TranslRel($\mathcal{A}$) reduces s_1 to s_2. Let t be a translation in $\mathcal{B}$ from s_1 into s_2. Suppose $\mathcal{P}[t, s_1, s_2]$. Let f be a function. If f is an elementary translation in $\mathcal{B}$ from s_2 into s_3, then $\mathcal{P}[f \cdot t, s_1, s_3]$.

The following propositions are true:

(21) Let A_1, A_2 be non-empty algebras over S and h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2. Let o be an operation symbol of S and i be a natural number. Suppose $i \in \operatorname{dom} \mathrm{Arity}(o)$. Let a be an element of $\mathrm{Args}(o, A_1)$. Then h(the result sort of $o) \cdot o_i^{A_1}(a, -) = o_i^{A_2}(h\#a, -) \cdot h(\mathrm{Arity}(o)_i)$.

(22) Let h be an endomorphism of A, o be an operation symbol of S, and i be a natural number. Suppose $i \in \operatorname{dom} \mathrm{Arity}(o)$. Let a be an element of $\mathrm{Args}(o, A)$. Then h(the result sort of $o) \cdot o_i^A(a, -) = o_i^A(h\#a, -) \cdot h(\mathrm{Arity}(o)_i)$.

(23) Let A_1, A_2 be non-empty algebras over S and h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2. Let s_1, s_2 be sort symbols of S and t be a function. Suppose t is an elementary translation in A_1 from s_1 into s_2. Then there exists a function T from (the sorts of $A_2)(s_1)$ into (the sorts of $A_2)(s_2)$ such that T is an elementary translation in A_2 from s_1 into s_2 and $T \cdot h(s_1) = h(s_2) \cdot t$.

(24) Let h be an endomorphism of A, s_1, s_2 be sort symbols of S, and t be a function. Suppose t is an elementary translation in A from s_1 into s_2. Then there exists a function T from (the sorts of $A)(s_1)$ into (the sorts of $A)(s_2)$ such that T is an elementary translation in A from s_1 into s_2 and $T \cdot h(s_1) = h(s_2) \cdot t$.

(25) Let A_1, A_2 be non-empty algebras over S and h be a many sorted function from A_1 into A_2. Suppose h is a homomorphism of A_1 into A_2. Let s_1, s_2 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let t be a translation in A_1 from s_1 into s_2. Then there exists a translation T in A_2 from s_1 into s_2 such that $T \cdot h(s_1) = h(s_2) \cdot t$.

(26) Let h be an endomorphism of A and s_1, s_2 be sort symbols of S. Suppose TranslRel(S) reduces s_1 to s_2. Let t be a translation in A from s_1 into s_2. Then there exists a translation T in A from s_1 into s_2 such that $T \cdot h(s_1) = h(s_2) \cdot t$.

2. Compatibility, invariantness, and stability

Let S be a non empty non void many sorted signature, let A be an algebra over S, and let R be a many sorted relation indexed by A. We say that R is compatible if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let o be an operation symbol of S and a, b be functions. Suppose $a \in \mathrm{Args}(o,A)$ and $b \in \mathrm{Args}(o,A)$ and for every natural number n such that $n \in \mathrm{dom}\,\mathrm{Arity}(o)$ holds $\langle a(n), b(n)\rangle \in R(\mathrm{Arity}(o)_n)$. Then $\langle(\mathrm{Den}(o,A))(a), (\mathrm{Den}(o,A))(b)\rangle \in R(\text{the result sort of } o)$.

We say that R is invariant if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let s_1, s_2 be sort symbols of S and t be a function. Suppose t is an elementary translation in A from s_1 into s_2. Let a, b be sets. If $\langle a, b\rangle \in R(s_1)$, then $\langle t(a), t(b)\rangle \in R(s_2)$.

We say that R is stable if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let h be an endomorphism of A, s be a sort symbol of S, and a, b be sets. If $\langle a, b\rangle \in R(s)$, then $\langle h(s)(a), h(s)(b)\rangle \in R(s)$.

The following propositions are true:

(27) Let R be an equivalence many sorted relation indexed by A. Then R is compatible if and only if R is a congruence of A.

(28) Let R be a many sorted relation indexed by A. Then R is invariant if and only if for all sort symbols s_1, s_2 of S such that TranslRel(S) reduces s_1 to s_2 and for every translation f in A from s_1 into s_2 and for all sets a, b such that $\langle a, b\rangle \in R(s_1)$ holds $\langle f(a), f(b)\rangle \in R(s_2)$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Observe that every equivalence many sorted relation indexed by A which is invariant is also compatible and every equivalence many sorted relation indexed by A which is compatible is also invariant.

Let X be a non empty set. One can verify that id_X is non empty.

Now we present two schemes. The scheme *MSRExistence* deals with a non empty set $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a many sorted relation R indexed by $\mathcal{B}$ such that for every element i of $\mathcal{A}$ and for all elements a, b of $\mathcal{B}(i)$ holds $\langle a, b\rangle \in R(i)$ if and only if $\mathcal{P}[i,a,b]$

for all values of the parameters.

The scheme *MSRLambdaU* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

(i) There exists a many sorted relation R indexed by $\mathcal{B}$ such that for every set i such that $i \in \mathcal{A}$ holds $R(i) = \mathcal{F}(i)$, and

(ii) for all many sorted relations R_1, R_2 indexed by $\mathcal{B}$ such that for every set i such that $i \in \mathcal{A}$ holds $R_1(i) = \mathcal{F}(i)$ and for every set i such that $i \in \mathcal{A}$ holds $R_2(i) = \mathcal{F}(i)$ holds $R_1 = R_2$

provided the parameters satisfy the following condition:

- For every set i such that $i \in \mathcal{A}$ holds $\mathcal{F}(i)$ is a relation between $\mathcal{B}(i)$ and $\mathcal{B}(i)$.

Let I be a set and let A be a many sorted set indexed by I. The functor $\mathrm{id}(I,A)$ yielding a many sorted relation indexed by A is defined by:

(Def. 10) For every set i such that $i \in I$ holds $(\mathrm{id}(I,A))(i) = \mathrm{id}_{A(i)}$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. Observe that every many sorted relation indexed by A which is equivalence is also non-empty.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. One can verify that there exists a many sorted relation indexed by A which is invariant, stable, and equivalence.

3. Invariant, stable, and invariant stable closure

In the sequel S is a non empty non void many sorted signature, A is a non-empty algebra over S, and R is a many sorted relation indexed by the sorts of A.

The scheme *MSRelCl* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, many sorted relations $\mathcal{Q}$, $\mathcal{D}$ indexed by $\mathcal{B}$, a unary predicate $\mathcal{Q}$, and a ternary predicate $\mathcal{P}$, and states that:

$\mathcal{Q}[\mathcal{D}]$ and $\mathcal{Q} \subseteq \mathcal{D}$ and for every many sorted relation P indexed by $\mathcal{B}$ such that $\mathcal{Q}[P]$ and $\mathcal{Q} \subseteq P$ holds $\mathcal{D} \subseteq P$

provided the parameters satisfy the following conditions:

- Let R be a many sorted relation indexed by $\mathcal{B}$. Then $\mathcal{Q}[R]$ if and only if for all sort symbols s_1, s_2 of $\mathcal{A}$ and for every function f from (the sorts of $\mathcal{B})(s_1)$ into (the sorts of $\mathcal{B})(s_2)$ such that $\mathcal{P}[f,s_1,s_2]$ and for all sets a, b such that $\langle a, b\rangle \in R(s_1)$ holds $\langle f(a), f(b)\rangle \in R(s_2)$,
- Let s_1, s_2, s_3 be sort symbols of $\mathcal{A}$, f_1 be a function from (the sorts of $\mathcal{B})(s_1)$ into (the sorts of $\mathcal{B})(s_2)$, and f_2 be a function from (the sorts of $\mathcal{B})(s_2)$ into (the sorts of $\mathcal{B})(s_3)$. If $\mathcal{P}[f_1,s_1,s_2]$ and $\mathcal{P}[f_2,s_2,s_3]$, then $\mathcal{P}[f_2 \cdot f_1,s_1,s_3]$,
- For every sort symbol s of $\mathcal{A}$ holds $\mathcal{P}[\mathrm{id}_{(\text{the sorts of } \mathcal{B})(s)},s,s]$, and
- Let s be a sort symbol of $\mathcal{A}$ and a, b be elements of $\mathcal{B}$, s. Then $\langle a, b\rangle \in \mathcal{D}(s)$ if and only if there exists a sort symbol s' of $\mathcal{A}$ and there exists a function f from (the sorts of $\mathcal{B})(s')$ into (the sorts of $\mathcal{B})(s)$ and there exist elements x, y of $\mathcal{B}$, s' such that $\mathcal{P}[f,s',s]$ and $\langle x, y\rangle \in \mathcal{Q}(s')$ and $a = f(x)$ and $b = f(y)$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation indexed by the sorts of A. The functor $\mathrm{InvCl}(R)$ is an invariant many sorted relation indexed by A and is defined by:

(Def. 11) $R \subseteq \mathrm{InvCl}(R)$ and for every invariant many sorted relation Q indexed by A such that $R \subseteq Q$ holds $\mathrm{InvCl}(R) \subseteq Q$.

Next we state two propositions:

(29) Let R be a many sorted relation indexed by the sorts of A, s be a sort symbol of S, and a, b be elements of A, s. Then $\langle a, b\rangle \in (\mathrm{InvCl}(R))(s)$ if and only if there exists a sort symbol s' of S and there exist elements x, y of A, s' and there exists a translation t in A from s' into s such that $\mathrm{TranslRel}(S)$ reduces s' to s and $\langle x, y\rangle \in R(s')$ and $a = t(x)$ and $b = t(y)$.

(30) For every stable many sorted relation R indexed by A holds $\mathrm{InvCl}(R)$ is stable.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation indexed by the sorts of A. The functor $\mathrm{StabCl}(R)$ is a stable many sorted relation indexed by A and is defined by:

(Def. 12) $R \subseteq \mathrm{StabCl}(R)$ and for every stable many sorted relation Q indexed by A such that $R \subseteq Q$ holds $\mathrm{StabCl}(R) \subseteq Q$.

The following two propositions are true:

(31) Let R be a many sorted relation indexed by the sorts of A, s be a sort symbol of S, and a, b be elements of A, s. Then $\langle a, b\rangle \in (\mathrm{StabCl}(R))(s)$ if and only if there exist elements x, y of A, s and there exists an endomorphism h of A such that $\langle x, y\rangle \in R(s)$ and $a = h(s)(x)$ and $b = h(s)(y)$.

(32) $\mathrm{InvCl}(\mathrm{StabCl}(R))$ is stable.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation indexed by the sorts of A. The functor $\mathrm{TRS}(R)$ is an invariant stable many sorted relation indexed by A and is defined by:

(Def. 13) $R \subseteq \mathrm{TRS}(R)$ and for every invariant stable many sorted relation Q indexed by A such that $R \subseteq Q$ holds $\mathrm{TRS}(R) \subseteq Q$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a non-empty many sorted relation indexed by A. One can check the following observations:

- $\mathrm{InvCl}(R)$ is non-empty,
- $\mathrm{StabCl}(R)$ is non-empty, and
- $\mathrm{TRS}(R)$ is non-empty.

We now state several propositions:

(33) For every invariant many sorted relation R indexed by A holds $\mathrm{InvCl}(R) = R$.

(34) For every stable many sorted relation R indexed by A holds $\mathrm{StabCl}(R) = R$.

(35) For every invariant stable many sorted relation R indexed by A holds $\mathrm{TRS}(R) = R$.

(36) $\mathrm{StabCl}(R) \subseteq \mathrm{TRS}(R)$ and $\mathrm{InvCl}(R) \subseteq \mathrm{TRS}(R)$ and $\mathrm{StabCl}(\mathrm{InvCl}(R)) \subseteq \mathrm{TRS}(R)$.

(37) $\mathrm{InvCl}(\mathrm{StabCl}(R)) = \mathrm{TRS}(R)$.

(38) Let R be a many sorted relation indexed by the sorts of A, s be a sort symbol of S, and a, b be elements of A, s. Then $\langle a, b\rangle \in (\mathrm{TRS}(R))(s)$ if and only if there exists a sort symbol s' of S such that $\mathrm{TranslRel}(S)$ reduces s' to s and there exist elements l, r of A, s' and there exists an endomorphism h of A and there exists a translation t in A from s' into s such that $\langle l, r\rangle \in R(s')$ and $a = t(h(s')(l))$ and $b = t(h(s')(r))$.

4. Equational theory

We now state four propositions:

(39) Let A be a set and R, E be binary relations on A. Suppose that for all sets a, b such that $a \in A$ and $b \in A$ holds $\langle a, b\rangle \in E$ iff a and b are convertible w.r.t. R. Then E is total, symmetric, and transitive.

(40) Let A be a set, R be a binary relation on A, and E be an equivalence relation of A. Suppose $R \subseteq E$. Let a, b be sets. If $a \in A$ and $b \in A$ and a and b are convertible w.r.t. R, then $\langle a, b\rangle \in E$.

(41) Let A be a non empty set, R be a binary relation on A, and a, b be elements of A. Then $\langle a, b\rangle \in \mathrm{EqCl}(R)$ if and only if a and b are convertible w.r.t. R.

(42) Let S be a non empty set, A be a non-empty many sorted set indexed by S, R be a many sorted relation indexed by A, s be an element of S, and a, b be elements of $A(s)$. Then $\langle a, b\rangle \in (\mathrm{EqCl}(R))(s)$ if and only if a and b are convertible w.r.t. $R(s)$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S. An equational theory of A is a stable invariant equivalence many sorted relation indexed by A. Let R be a many sorted relation indexed by A. The functor $\mathrm{EqCl}(R,A)$ yields an equivalence many sorted relation indexed by A and is defined as follows:

(Def. 14) $\mathrm{EqCl}(R,A) = \mathrm{EqCl}(R)$.

We now state four propositions:

(43) For every many sorted relation R indexed by A holds $R \subseteq \mathrm{EqCl}(R,A)$.

(44) Let R be a many sorted relation indexed by A and E be an equivalence many sorted relation indexed by A. If $R \subseteq E$, then $\mathrm{EqCl}(R,A) \subseteq E$.

(45) Let R be a stable many sorted relation indexed by A, s be a sort symbol of S, and a, b be elements of A, s. Suppose a and b are convertible w.r.t. $R(s)$. Let h be an endomorphism of A. Then $h(s)(a)$ and $h(s)(b)$ are convertible w.r.t. $R(s)$.

(46) For every stable many sorted relation R indexed by A holds $\mathrm{EqCl}(R,A)$ is stable.

Let us consider S, A and let R be a stable many sorted relation indexed by A. Observe that $\mathrm{EqCl}(R,A)$ is stable.

We now state two propositions:

(47) Let R be an invariant many sorted relation indexed by A, s_1, s_2 be sort symbols of S, and a, b be elements of A, s_1. Suppose a and b are convertible w.r.t. $R(s_1)$. Let t be a function. Suppose t is an elementary translation in A from s_1 into s_2. Then $t(a)$ and $t(b)$ are convertible w.r.t. $R(s_2)$.

(48) For every invariant many sorted relation R indexed by A holds $\mathrm{EqCl}(R,A)$ is invariant.

Let us consider S, A and let R be an invariant many sorted relation indexed by A. Note that $\mathrm{EqCl}(R,A)$ is invariant.

The following propositions are true:

(49) Let S be a non empty set, A be a non-empty many sorted set indexed by S, and R, E be many sorted relations indexed by A. Suppose that for every element s of S and for all elements a, b of $A(s)$ holds $\langle a, b\rangle \in E(s)$ iff a and b are convertible w.r.t. $R(s)$. Then E is equivalence.

(50) Let R, E be many sorted relations indexed by A. Suppose that for every sort symbol s of S and for all elements a, b of A, s holds $\langle a, b\rangle \in E(s)$ iff a and b are convertible w.r.t. $(\mathrm{TRS}(R))(s)$. Then E is an equational theory of A.

(51) Let S be a non empty set, A be a non-empty many sorted set indexed by S, R be a many sorted relation indexed by A, and E be an equivalence many sorted relation indexed by A. Suppose $R \subseteq E$. Let s be an element of S and a, b be elements of $A(s)$. If a and b are convertible w.r.t. $R(s)$, then $\langle a, b\rangle \in E(s)$.

Let S be a non empty non void many sorted signature, let A be a non-empty algebra over S, and let R be a many sorted relation indexed by the sorts of A. The functor $\mathrm{EqTh}(R)$ is an equational theory of A and is defined by:

(Def. 15) $R \subseteq \mathrm{EqTh}(R)$ and for every equational theory Q of A such that $R \subseteq Q$ holds $\mathrm{EqTh}(R) \subseteq Q$.

One can prove the following propositions:

(52) For every many sorted relation R indexed by A holds $\mathrm{EqCl}(R,A) \subseteq \mathrm{EqTh}(R)$ and $\mathrm{InvCl}(R) \subseteq \mathrm{EqTh}(R)$ and $\mathrm{StabCl}(R) \subseteq \mathrm{EqTh}(R)$ and $\mathrm{TRS}(R) \subseteq \mathrm{EqTh}(R)$.

(53) Let R be a many sorted relation indexed by A, s be a sort symbol of S, and a, b be elements of A, s. Then $\langle a, b\rangle \in (\mathrm{EqTh}(R))(s)$ if and only if a and b are convertible w.r.t. $(\mathrm{TRS}(R))(s)$.

(54) For every many sorted relation R indexed by A holds $\mathrm{EqTh}(R) = \mathrm{EqCl}(\mathrm{TRS}(R),A)$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[3] Grzegorz Bancerek. Reduction relations. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/rewrite1.html`.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[5] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/funct_7.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[8] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[9] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[10] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_3.html`.

[11] Małgorzata Korolkiewicz. Many sorted quotient algebra. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_4.html`.

[12] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pralg_1.html`.

[13] Robert Milewski. Lattice of congruences in many sorted algebra. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/msualg_5.html`.

[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[15] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[16] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_1.html`.

[17] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[19] Wolfgang Wechler. *Universal Algebra for Computer Scientists*, volume 25 of *EATCS Monographs on TCS*. Springer–Verlag, 1992.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[21] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

[22] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_2.html`.

Received February 9, 1996

Published January 2, 2004